- بسم الله الرحمن الرحيم -

CALCULUS

Textbook: Calculus / Finney and Thomas, 8th edition, 1990.

Reference: Calculus / James Stewart, 10th edition, 2003.

Syllabus:

1- Prerequisites for calculus:
  - Numbers and sets.
  - Coordinates in the plane.
  - Slope and equations for lines.
  - Functions (Intervals, Domain, Range, and graph)
  - Shifting of graph
  - Even and odd functions.
  - Conic sections.
  - Trigonometric Functions.
  - Absolute values and inequalities.

2- Limits and continuity.

3- Derivatives:
  - Differential rules.
  - Velocity, speed, and acceleration.
  - Derivative of trig. functions.
  - Chain rule.
  - Implicit differentiation.
  - L'Hopital rule.
  - Parametric equations.

4- Applications of derivatives.
  - Related rates.
  - Maxima, minima, and mean value theorem.
  - Graphing of functions.
  - Optimization.

5- Integration:
- Antiderivatives.
- Rules of integral.
- First and second fundamental theorems.
- Integration by substitution.
- Numerical integration (Trapezoidal, midpoint, and Simpson's rule.

6- Applications of definite integrals:
- Area between curves.
- Volumes of solids.
- Length of curves.
- Surface area of solids.

7- Transcendental functions:
- Inverse functions.
- Logarithmic functions.
- Exponential functions.
- Inverse trig. functions.

8- Techniques of integration:
- Integration by parts.
- Trig. Integrals.
- Trig. substitutions.
- Integration of rational functions (Partial fractions).
- Improper integrals.

9- Matrices, determinants, and Cramer's rule.

What is calculus?

Calculus is a study of change.

Also we can define Calculus as the part of mathematics that deals with limits.

Calculus was first created to meet the mathematical needs of the scientists of the 17th century.

In 17th century; Isaac Newton invented his version of Calculus in order to explain the motion of the planets around the sun.

Today, Calculus is used in:

- Calculating the orbits of satellites and space crafts.
- Predicting population size.
- Forecasting the weather.
- Forecasting the global trends (economy).
- Designing x-ray and ultra-sound equipments.

This list is practically endless.

In engineering, Calculus has wide applications.

Briefly, every field today uses Calculus in some way.

CHAPTER ONE

PREREQUISITES FOR CALCULUS

Numbers and Sets :

① **Numbers:** We can classify the numbers into :

1- Natural numbers : 0 , 1 , 2 , 3 , 4 , ...

or 1 , 2 , 3 , 4 , ...

(No agreement about the number 0) .

2- Integer numbers: ... , −3 , −2 , −1 , 0 , 1 , 2 , 3 , ...

3- Rational numbers : The numbers that can be written as $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

4- Irrational numbers : For example , $\sqrt{3}$, π , and 0.54687

5- Real numbers (R) : All the measuring numbers.

② **Sets :** A set is a collection of distinct elements.

There are two ways to describe the set :

1- By definition :

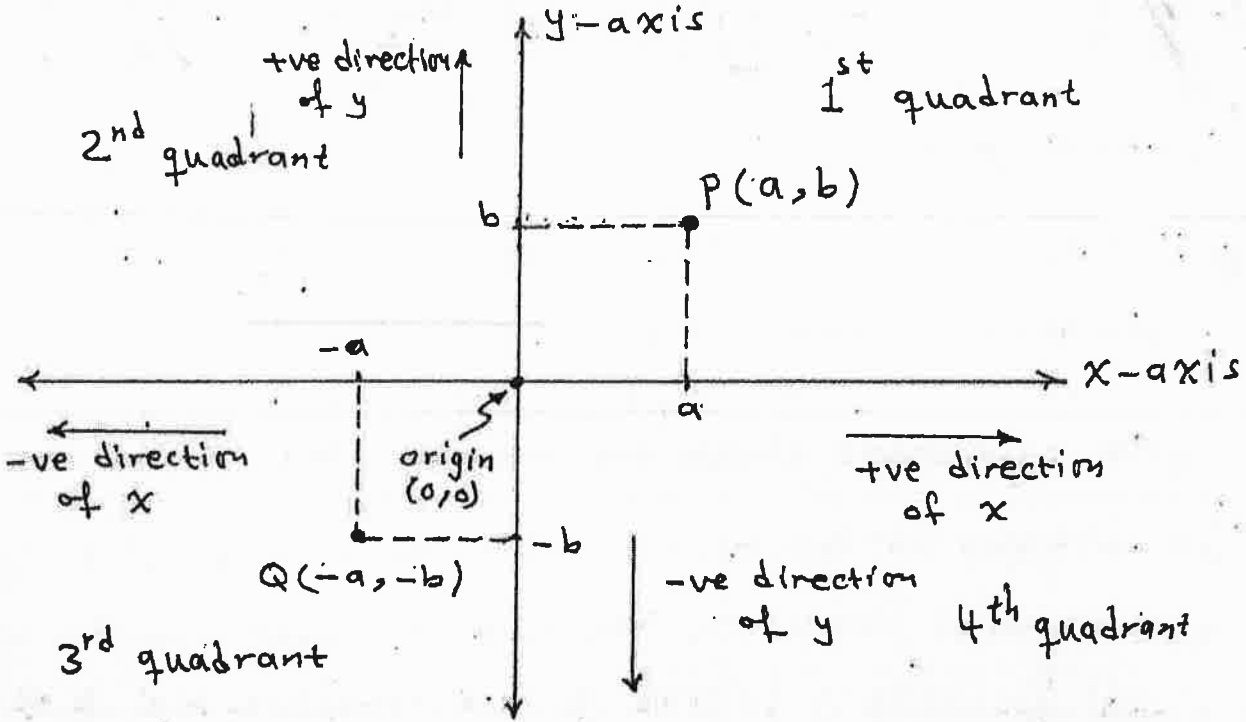
For example : A is a set of colors of Iraqi flag .

2- By braces $\{ \ \}$:

For example : $A = \{$Red , White , Green , Black$\}$.

Subset : If every element of set A is also an element of set B , then A is said subset of B .

1.1: Coordinates and Graphs in the Plane :



For the point P :

(a, b) : coordinate pair.
a : x- coordinate.
b : y- coordinate.

The point P is the point where the perpendicular to the x-axis at a crosses the perpendicular to the y-axis at b.

Notes ①: The scale in one axis must be uniform.
②. We do not need to use the same scale on the two axes.

Distance between points:

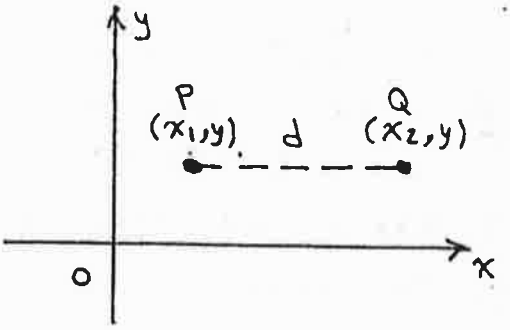
There are three cases:

① Horizontal distance.
② Vertical distance.
③ Oblique distance.

① Horizontal distance:

The distance d is : $\boxed{d = x_2 - x_1}$

Where: x_2 = the x-coordinate for the right hand point (Q).
x_1 = the x-coordinate for the left hand point (P).

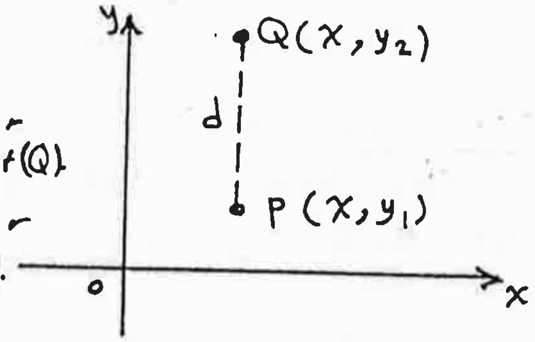


Ex.: Find the distance between (-2,3) and (4,3).

sol. $d = x_2 - x_1 = 4-(-2) = 6$ units

② Vertical distance:

$\boxed{d = y_2 - y_1}$ Where: y_2 = y-coordinate for the upper point (Q).
y_1 = y-coordinate for the lower point (P).



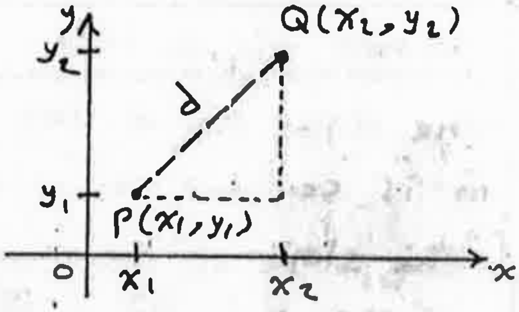
Ex. Find the distance between (3,-4) and (3,-1).

sol. $d = y_2 - y_1 = -1-(-4) = -1+4 = 3$ units.

③ Oblique distance:

Use Pythagorean theorem:

$$\boxed{d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}}$$



Ex. Find the distance between (-1,2) & (3,4).

sol. $d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} = \sqrt{(3-(-1))^2 + (4-2)^2} = \sqrt{20} = 2\sqrt{5}$ units

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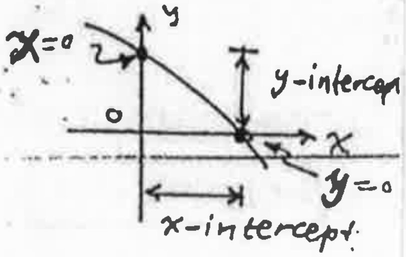
Intercepts: The points where the graph crosses the axes are called intercepts.

x-intercept is the x-coordinate of the graph at y=0.
y-intercept is the y-coordinate of the graph at x=0.



Ex. Find the intercepts of the line $y = x+3$.

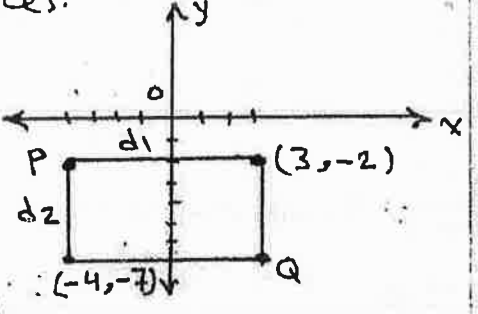
sol. For x-intercept, $y=0 \Rightarrow 0 = x+3 \Rightarrow x=-3$
For y-intercept, $x=0 \Rightarrow y=0+3 \Rightarrow y=3$
$\Rightarrow x=-3$ and $y=3$ are the intercepts.

Exercises 1.1 / P.8 :

㊳ A rectangle with sides parallel to the axes has vertices at $(3,-2)$ and $(-4,-7)$. Find :

ⓐ The coordinates of the other two vertices.

ⓑ The area of the rectangle.



Sol ⓐ Since the sides parallel to the axes

$\Rightarrow$ P is $(-4,-2)$ and Q is $(3,-7)$.

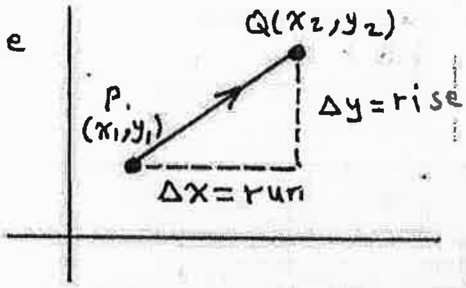
ⓑ $d_1 = 3-(-4) = 7$, $d_2 = -2-(-7) = 5$

$\Rightarrow$ Area of rectangle $= d_1 * d_2 = 7*5 = \boxed{35 \text{ units}}$

1.2: Slope and Equations for Lines:

Increments (net changes) : When a particle moves from p to Q , the increments Δx and Δy are : $\Delta x = x_2 - x_1$

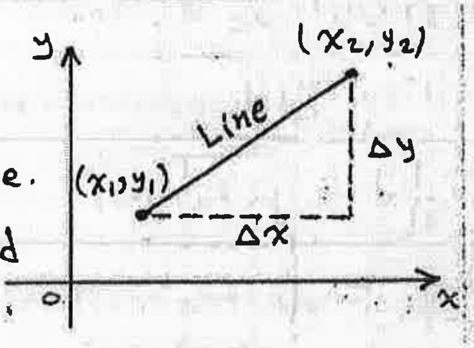
$\Delta y = y_2 - y_1$



Ex. Find the increments if a particle moves from $(4,-3)$ to $(2,5)$.

Sol. $\Delta x = 2-4 = -2$ & $\Delta y = 5-(-3) = 8$

Slope of a line :



The slope of a line $= m = \dfrac{\Delta y}{\Delta x} = \dfrac{y_2-y_1}{x_2-x_1}$

m is constant value in every point of the line.

* The slope of vertical line is not defined since $m = \dfrac{\Delta y}{\Delta x} = \dfrac{\Delta y}{0} = \infty$

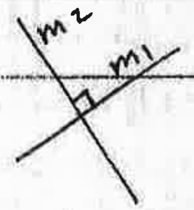
* The slope of horizontal line is zero since $m = \dfrac{\Delta y}{\Delta x} = \dfrac{0}{\Delta x} = 0$

* The slopes of parallel lines are equal.

$\boxed{m_1 = m_2}$

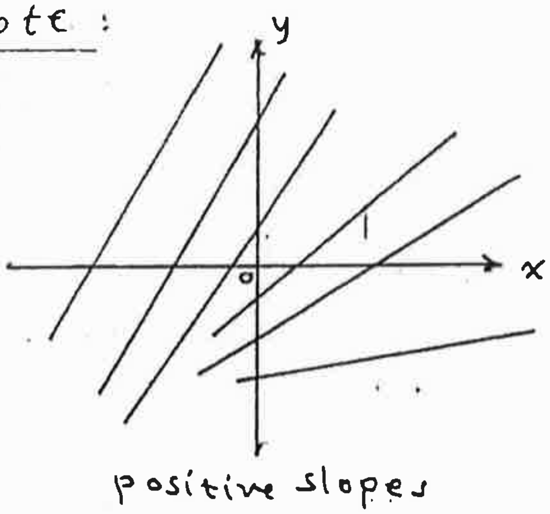
* The slopes of perpendicular lines are:

$\boxed{m_1 * m_2 = -1}$ or $\boxed{m_1 = \dfrac{-1}{m_2}}$ or $\boxed{m_2 = \dfrac{-1}{m_1}}$

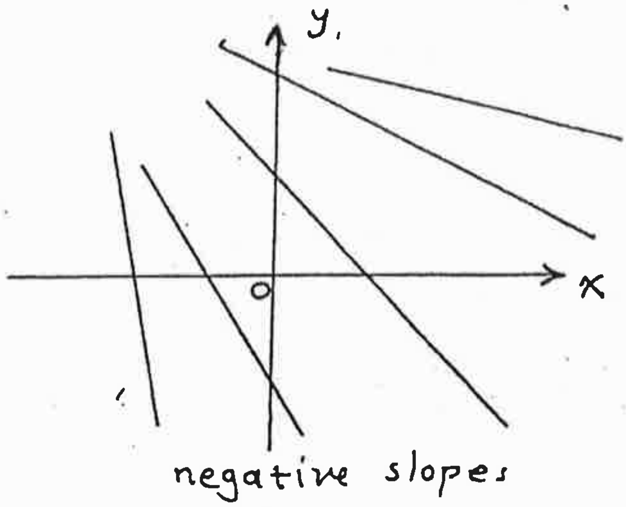


For example if $m_1 = -3 \Rightarrow m_2 = \dfrac{1}{3}$

Note :



positive slopes



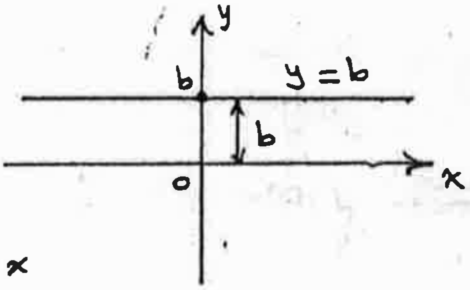
negative slopes

Equations for Lines :

An equation for a line is an equation that is satisfied by the coordinates of every point on the line but is not satisfied by the coordinates of points that lie elsewhere.

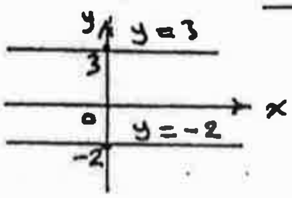
Horizontal and vertical lines:

The equation for horizontal line is $\boxed{y = b}$.

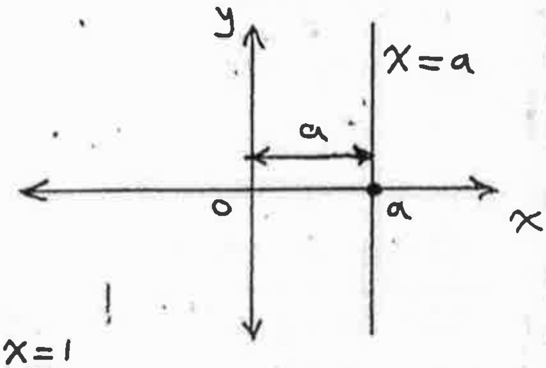


For example : ① $y = 3$

② $y = -2$

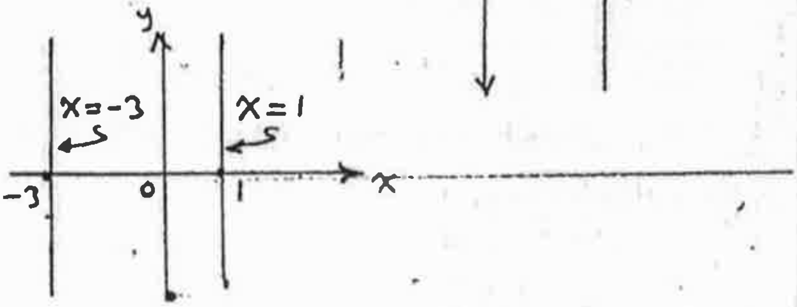


The equation for vertical line is $\boxed{x = a}$

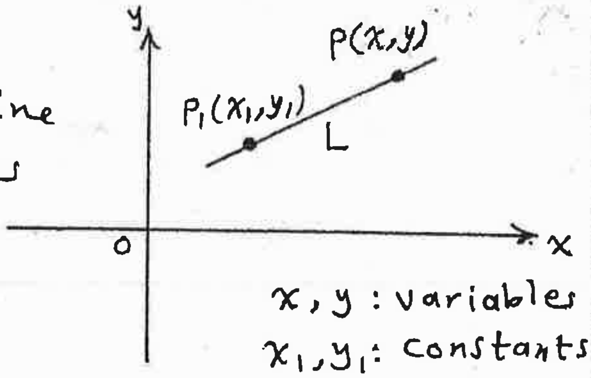


For example : ① $x = 1$

$x = -3$



Point – slope equation:



x , y : variables
x_1, y_1: constants

To find the equation for the line L with given slope(m) and passes through the point $P_1(x_1, y_1)$:

$$m = \frac{\Delta y}{\Delta x} \Longrightarrow m = \frac{y - y_1}{x - x_1}$$

$$\Longrightarrow \boxed{y - y_1 = m(x - x_1)}$$ — point-slope equation

Ex.1: Write the equation for the line that passes through the point (-2, 3) with slope 4.

sol.:
$$y - y_1 = m(x - x_1)$$
$$y - 3 = 4(x - (-2)) \Longrightarrow y - 3 = 4(x+2)$$
$$\Longrightarrow y - 3 = 4x + 8 \Longrightarrow y = 4x + 11$$

Ex.2: Write the equation for the line passes through the points (-2, -1) and (3, 4).

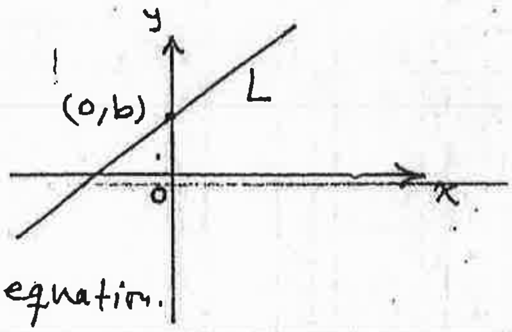
sol.
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - (-1)}{3 - (-2)} = \frac{5}{5} = 1$$

Now, we have m=1 with any of the two points.

Take the point (-2, -1):

$$\Longrightarrow y - y_1 = m(x - x_1)$$
$$y - (-1) = 1(x - (-2)) \Longrightarrow y + 1 = x + 2 \Longrightarrow y = x + 1$$

Slope - intercept equation:



To find the equation for the line L with given slope (m) and given y-intercept (b) :

$$y - y_1 = m(x - x_1) \Longrightarrow y - b = m(x - 0)$$

$$\Longrightarrow \boxed{y = mx + b}$$ — slope-intercept equation.

Ex.1: Find the equation for the line with slope -2 and y-intercept equals 3.

sol. $y = mx + b$, $m = -2$ & $b = 3$

$\Rightarrow y = -2x + 3$

Ex.2: Find the slope and the y-intercept for the line $8x + 5y = 20$.

sol. Write $8x + 5y = 20$ in the $y = mx + b$ form:

$\Rightarrow 5y = -8x + 20 \Rightarrow y = \frac{1}{5}(-8x + 20) \Rightarrow y = \frac{-8}{5}x + 4$

Compare $y = \frac{-8}{5}x + 4$ with $y = mx + b$

$\Rightarrow m = \frac{-8}{5}$ and $b = 4$

The distance from a point to a line:

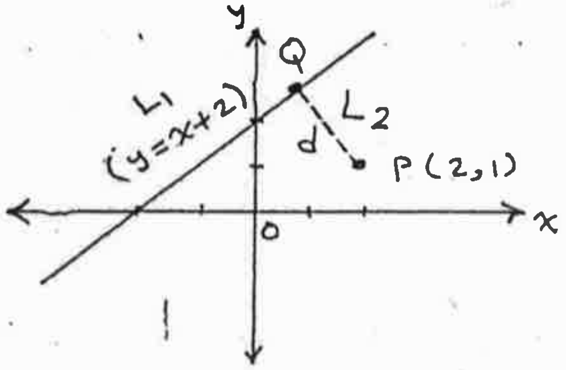
To find the distance from a point P to a line L_1:

1- Find the equation of the line L_2 that passes through the point P and perpendicular to the line L_1.

2- Find the point of intersection of two lines (Q).

3- Calculate the distance between the point P. to the point of intersection Q.

Ex. Find the distance from the point $P(2,1)$ and the line $y = x + 2$.



Sol. 1- $y = x + 2 \Rightarrow m_1 = 1$

$\Rightarrow$ slope of perpendicular line L_2 is $m_2 = \frac{-1}{m_1} = \frac{-1}{1} = -1$

$y - y_1 = m_2(x - x_1)$

$y - 1 = -1(x - 2) \Rightarrow y - 1 = -x + 2$

$\Rightarrow y = -x + 3$ eq. for L_2

2- To find the coordinates of intersection point Q:
y-coordinates and x-coordinates for L_1 & L_2 at the point Q are equal. (or $y_1 = y_2$ and $x_1 = x_2$).

Use $y_1 = y_2$

$\Rightarrow x+2 = -x+3 \Rightarrow 2x = 1 \Rightarrow x = \frac{1}{2}$

from any of the two equations find y:

Use the equation of $L_1 \Rightarrow y = x+2 = \frac{1}{2} + 2 = \frac{5}{2}$

$\Rightarrow$ The coordinates of Q is $(\frac{1}{2}, \frac{5}{2})$.

3- Distance from p to L_1 is the distance from the point p to the point Q:

$\Rightarrow d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$, $P(2,1)$ & $Q(\frac{1}{2}, \frac{5}{2})$

$\Rightarrow d = \sqrt{(\frac{1}{2}-2)^2 + (\frac{5}{2}-1)^2} = \sqrt{\frac{18}{4}} = \frac{3}{2}\sqrt{2}$ units.

The general equation for the line :

The general equation for the line is $\boxed{ax + by + c = 0}$ where a, b, and c are constants.

For example $3x + 2y - 5 = 0$.

Exercises 1.2 / P.18 :

⑤ Find the net changes Δx and Δy in the particle coordinates if the particle moves from $A(-3,1)$ to $B(-8,1)$.

sol. $\Delta x = x_2 - x_1 = -8 - (-3) = -5$.

$\Delta y = y_2 - y_1 = 1 - 1 = 0$.

㉑ Find the equation for the line that passes through the point $(-1,1)$ and has slope equals 1.

sol. $y - y_1 = m(x - x_1) \Rightarrow y - 1 = 1(x-(-1))$ ~~$\Rightarrow y - 1 = x + 1$~~

$\Rightarrow y = x + 2$

(27) Find the equation for the line that passes through the points (1,1) and (1,2).

sol. $m = \frac{\Delta y}{\Delta x} = \frac{y_2-y_1}{x_2-x_1} = \frac{2-1}{1-1} = \frac{1}{0} = \infty$

⇒ the line is vertical

⇒ the equation is $x = a$

a = x-coordinate for any of the two points = 1

⇒ the equation is $x = 1$

(34) Find the equation for the line with slope equals $-\frac{1}{2}$ and y-intercept equals -3.

sol. $y = mx + b$

$y = -\frac{1}{2}x - 3$

(39) Find the x and y intercepts for the line $4x - 3y = 12$.

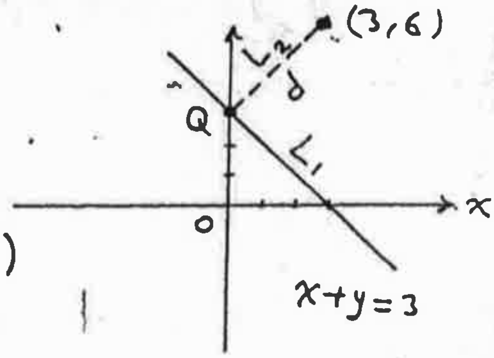
sol. For x-intercept, $y = 0 \Rightarrow 4x - 3(0) = 12 \Rightarrow x = \frac{12}{4} = 3$

⇒ x-intercept $= 3$;

For y-intercept, $x = 0 \Rightarrow 4(0) - 3y = 12 \Rightarrow y = \frac{12}{-3} = -4$

⇒ y-intercept $= -4$

(47) Find the distance from the point (3,6) and the line $x + y = 3$.



sol. $y = -x + 3 \Rightarrow m_1 = -1$

$\Rightarrow m_2 = \frac{-1}{m_1} = \frac{-1}{-1} = 1$

⇒ eq. of L_2 is: $y - y_1 = m(x - x_1)$

$y - 6 = 1(x - 3)$

$\Rightarrow y = x + 3$

To find Q:
$$\begin{array}{l} y = -x + 3 \\ y = x + 3 \\ \hline 2y = 6 \end{array} \quad \text{add} \qquad \longrightarrow y = 3$$

From L_2:

$y = x+3 \Rightarrow x = y-3 \Rightarrow x = 3-3 = 0$

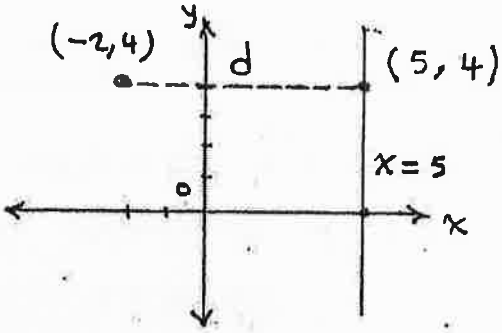
$\Rightarrow$ The coordinates of Q are $(0,3)$

$\Rightarrow d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = \sqrt{(3-0)^2+(6-3)^2} = \sqrt{18} = 3\sqrt{2}$ units

㊽ Find the distance from the point $(-2,4)$ and the line $x=5$.

sol. Since $x=5$ is vertical line, the distance is horizontal.

$\Rightarrow d = x_2 - x_1 = 5-(-2) = 7$ units



(51) Find the equation for the line that passes through the point $(1,0)$ and parallel to the line $2x+y=-2$.

sol. $2x+y=-2 \Rightarrow y=-2x-2 \Rightarrow m_1=-2$

The slope of the parallel lines are equal $[m_1=m_2]$

$\Rightarrow m_2=-2$, and the point is $(1,0)$ [given]

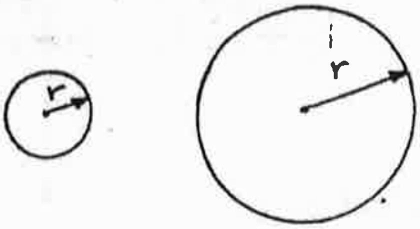
$\Rightarrow y-y_1 = m(x-x_1)$, $m=m_2$

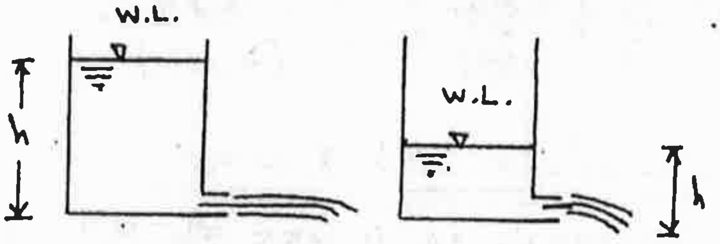
$\Rightarrow y-0 = -2(x-1)$

$\Rightarrow y = -2x+2$

1.3 : Functions and Their Graphs :

Function is a rule that relates inputs with outputs.





Ex.1 : $A = \pi r^2$

r = radius (inputs)

A = Area (outputs)

π = 3.14--- (constant)

Ex.2 : $P = \gamma h$

h = height (inputs)

P = pressure (outputs)

γ = density (constant)

Ex.3 : $y = x + 1$

x = inputs

y = outputs

Ex.4 : $y = \sqrt{x}$

x = inputs

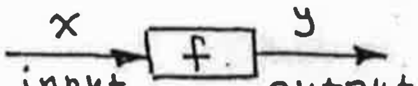
y = outputs

The function is denoted by f or g or h.
(Any symbol may be used).

For example : ① $y = f(x)$ y is a function of x.

② $A = g(r)$ A is a function of r.

The function is like a machine :

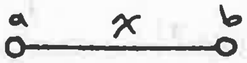


x is called independent variable.

y is called dependent variable (depends on x).

Intervals :

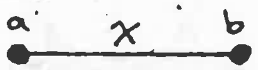
① Finite intervals :



open interval

symbol: $a < x < b$

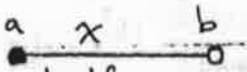
or (a, b)



closed interval

$a \leq x \leq b$

or $[a, b]$

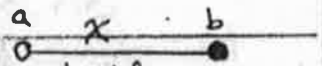


half-open

(or half-closed)

$a \leq x < b$

or. $[a, b)$



half-open

(or half-closed)

$a < x \leq b$

or $(a, b]$

② Infinite intervals:

$-\infty<x<\infty$	$x>a$	$x\geqslant a$	$x<b$	$x\leqslant b$
$(-\infty,\infty)$	(a,∞)	$[a,\infty)$	$(-\infty,b)$	$(-\infty,b]$

Domain and Range:

Domain is a set of allowable inputs.
Range is a set of outputs.

Restrictions in Domain:

1- Never divide by zero.
2- Value of square roots must be not negative.

Examples: Find the domain and range for the functions:

① $y=x^2$ ② $y=\frac{1}{x}$ ③ $y=\sqrt{1-x^2}$

Solutions: ① $y=x^2$

domain: all real numbers (or $-\infty<x<\infty$)
range: $y\geqslant 0$

② $y=\frac{1}{x}$

domain: $x\neq 0$
range: $y\neq 0$

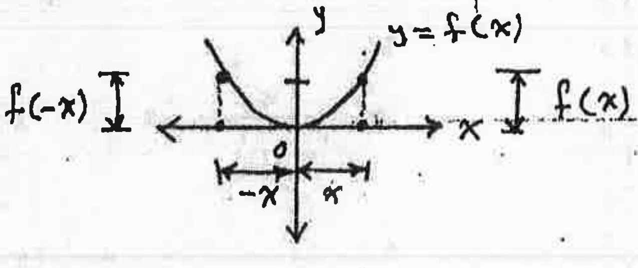
③ $y=\sqrt{1-x^2}$

domain: $-1\leqslant x\leqslant 1$
range: $0\leqslant y\leqslant 1$

Even and Odd Functions:

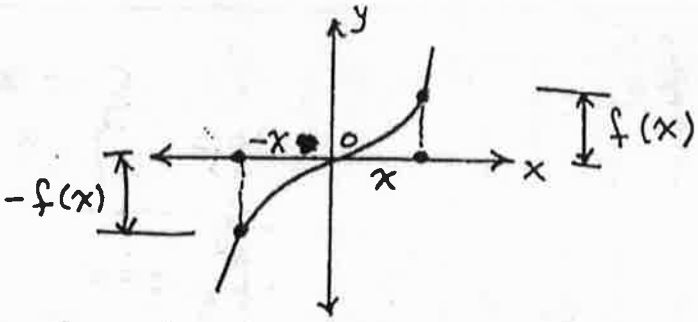
The function is even if $f(-x)=f(x)$.

this function is symmetric about y-axis.



The function is odd if $f(-x) = -f(x)$.

this function is symmetric about the origin $(0,0)$.



Ex.1: $y = x^2+1$

$f(-x) = (-x)^2+1 = x^2+1$

$f(x) = x^2+1$

$\Longrightarrow f(x) = f(-x) \Longrightarrow y = x^2+1$ is even function.

Ex.2: $y = x^3$

$f(-x) = (-x)^3 = -x^3$

$-f(x) = -x^3$

$-f(x) = f(-x) \Rightarrow y = x^3$ is odd function.

Note: The function may be neither even nor odd.

Integer-Valued Functions:

① $y = \lfloor x \rfloor$ integer floor for x.

Examples: $\lfloor 1.9 \rfloor = 1$, $\lfloor 1.2 \rfloor = 1$

$\lfloor -1.3 \rfloor = -2$, $\lfloor -0.5 \rfloor = -1$

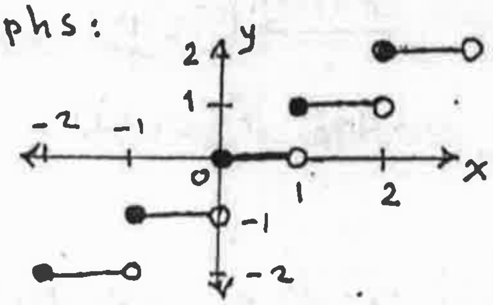
② $y = \lceil x \rceil$ integer ceiling for x.

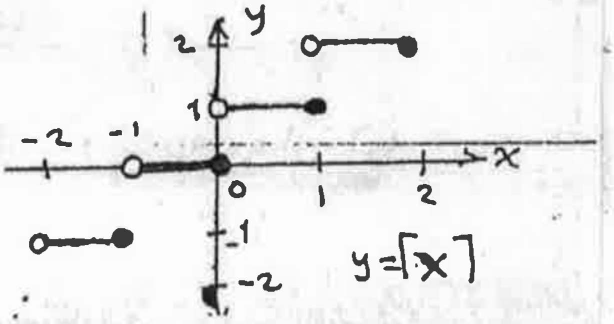
Examples: $\lceil 1.9 \rceil = 2$, $\lceil 1.2 \rceil = 2$

$\lceil -1.3 \rceil = -1$, $\lceil -0.5 \rceil = 0$

The graphs:

$y = \lfloor x \rfloor$

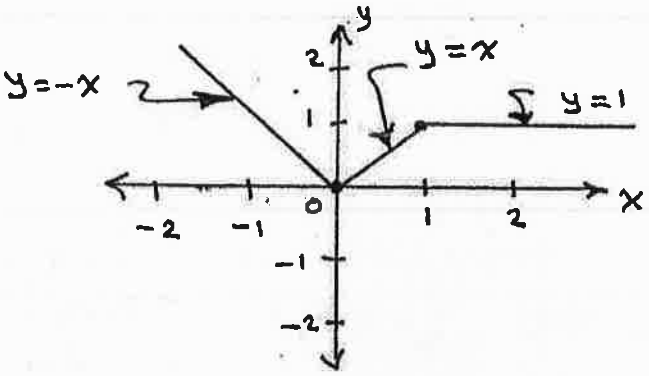




Functions Defined by Pieces :

Ex.

1. $$y = \begin{cases} -x & , x < 0 \\ x & , 0 \leq x \leq 1 \\ 1 & , x > 1 \end{cases}$$



this is three pieces function

Composition of Functions :

$$x \longrightarrow \boxed{g} \xrightarrow{g(x)} \boxed{f} \longrightarrow f(g(x))$$

Ex. If $g(x) = x^2$ and $f(x) = x - 7$, find $f(g(x))$.

sol. $f(g(x)) = x^2 - 7$

$g(f(x))$

$= (x-7)^2$

To find $f(g(2))$:

$g(2) = (2)^2 = 4$

$f(g(2)) = f(4) = 4 - 7 = \boxed{-3}$

or $f(g(2)) = (2)^2 - 7 = 4 - 7 = \boxed{-3}$

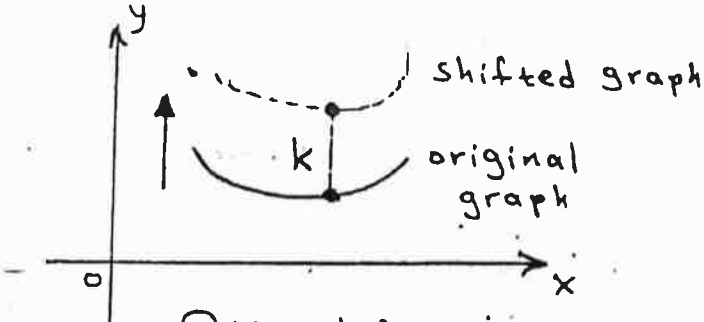
Notes ① The composition may be more than two functions

$$x \longrightarrow \boxed{g} \xrightarrow{g(x)} \boxed{f} \xrightarrow{f(g(x))} \boxed{h} \longrightarrow h(f(g(x)))$$

② The symbol $f \circ g$ means $f(g(x))$.

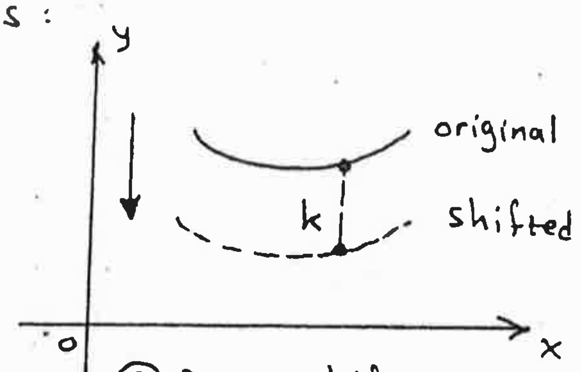
1.4: Shifts, Circles, and Parabolas:

Shifts: There are four cases:



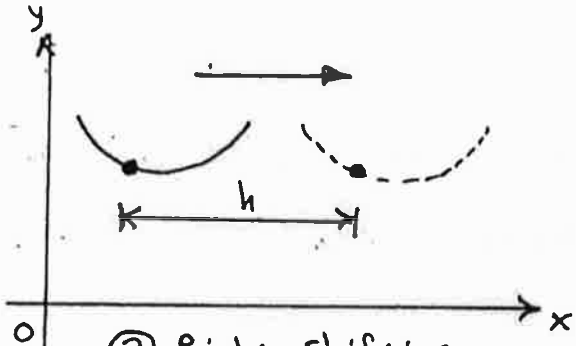
① Up shifting

$$\boxed{y-k=f(x)}$$



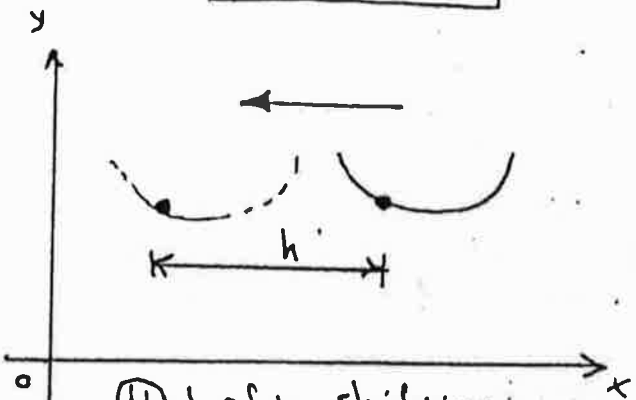
② Down shifting

$$\boxed{y+k=f(x)}$$



③ Right shifting

$$\boxed{y=f(x-h)}$$



④ Left shifting

$$\boxed{y=f(x+h)}$$

in which:

$f(x)$: original function

k : amount of vertical shifting (+ve only)

h : amount of horizontal shifting (+ve only).

Ex.1: For the function $y=x^2$, find the equation for the shifted graphs if:

(a) the graph shifted 3 units up.

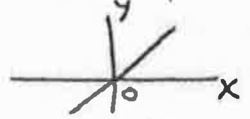
(b) " " " 2 units to the left.

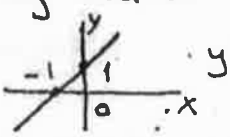
sol.: (a) $y-k=f(x) \Rightarrow y-3=x^2 \Rightarrow y=x^2+3$

(b) $y=f(x+h) \Longrightarrow y=(x+2)^2 \Rightarrow y=x^2+4x+4$

Note: The technique of shifting make the graph more easy.

$y=x$



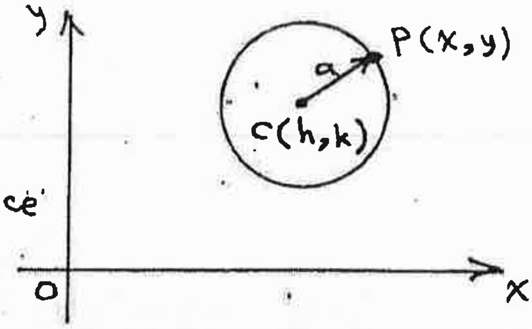


$y=x+1$

Conic Sections : The conic sections are :

① Circle ② Parabola ③ Hyperbola ④ Ellipse.

① Circle:



We have a circle with center c(h,k) and radius a.

From the formula of the distance between two points,(here c&P):

$\Rightarrow$ $\boxed{(x-h)^2+(y-k)^2=a^2}$ --- general equation for a circle.

If the center of the circle at the origin (0,0):

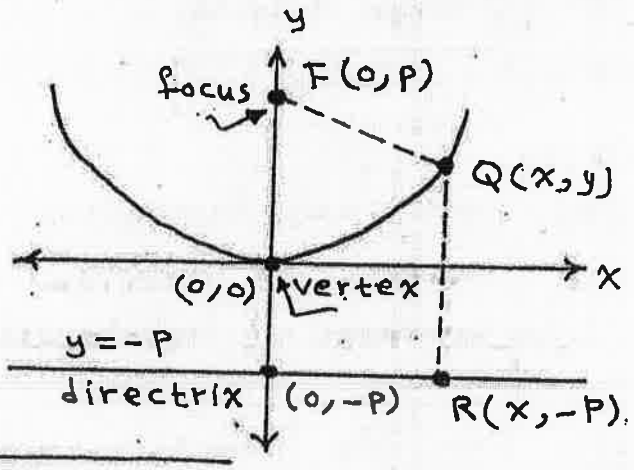
$\Rightarrow$ $\boxed{x^2+y^2=a^2}$ --- equation of circle centered at (0,0).

If the center at the origin and the radius = 1 unit:

$\Rightarrow$ $\boxed{x^2+y^2=1}$ --- the equation of unit circle.

② Parabola:

A parabola is a set of points in a plane that are equidistant from a given fixed point (focus) and fixed line (directrix).



$\overline{QF}=\overline{QR}$

$$\sqrt{(x-0)^2+(y-p)^2}=\sqrt{(x-x)^2+(y-(-p))^2}$$

$$\Rightarrow \sqrt{x^2+(y-p)^2}=\sqrt{(y+p)^2}$$

simplify: $\Rightarrow$ $\boxed{y=\frac{x^2}{4p}}$ --- equation for parabola with vertex at the origin and opens upward.

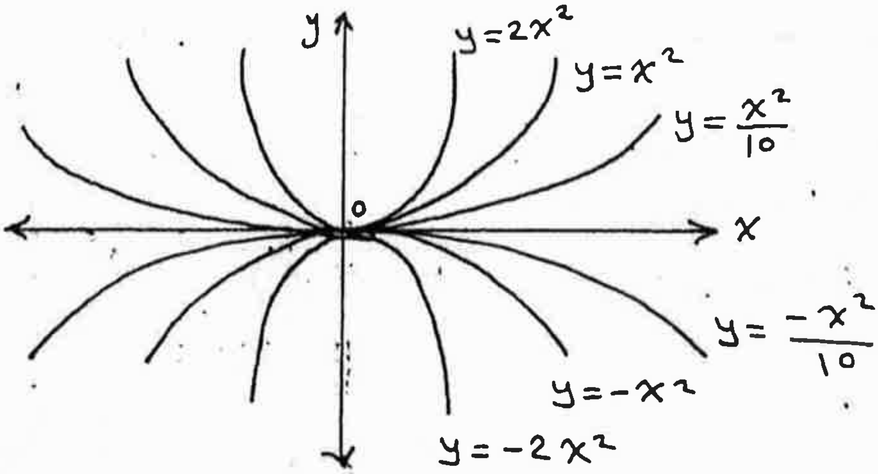
If the parabola opens downward:

$\Rightarrow$ $\boxed{y=-\frac{x^2}{4p}}$ -- equation for parabola with vertex at the origin and opens downward.

We can write the general equation for parabola as:

$\boxed{y = ax^2}$ in which $\boxed{a = \frac{1}{4p}}$ or $\boxed{a = -\frac{1}{4p}}$

Note: The parabola $y=ax^2$ widens as a approaches zero and narrows as a becomes large.

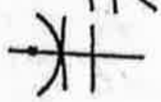


If the focus at the x-axis:

$\Rightarrow \boxed{x = ay^2}$

The parabola opens to the right if a is positive.

The parabola opens to the left if a is negative:



The Parabola $y=ax^2+bx+c$:

The general equation for parabola is $y=ax^2+bx+C$. This equation comes from shifting the equation $y=ax^2$ vertically and horizontally.

For example: If $y=ax^2$ is shifted up and right,

$\Rightarrow y-k=a(x-h)^2 \Rightarrow y=a(x^2-2xh+h^2)+k$.

$\Rightarrow y = ax^2 \underbrace{-2ah}_{b} x + \underbrace{ah^2+k}_{c}$

or $y=ax^2+bx+C$, where $b=-2ah$

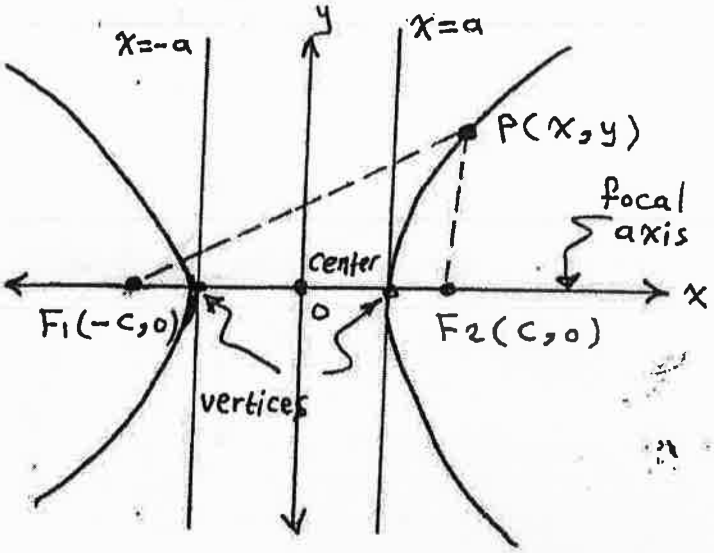
$c=ah^2+k$

③ Hyperbola :

A hyperbola is the set of points in a plane whose distances from two fixed points have a constant difference. The two fixed points are the foci of the hyperbola.

$\overline{PF_1} - \overline{PF_2} = 2a$.

This equation leads to the standard equations for the hyperbola.



$$\Rightarrow \boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$$

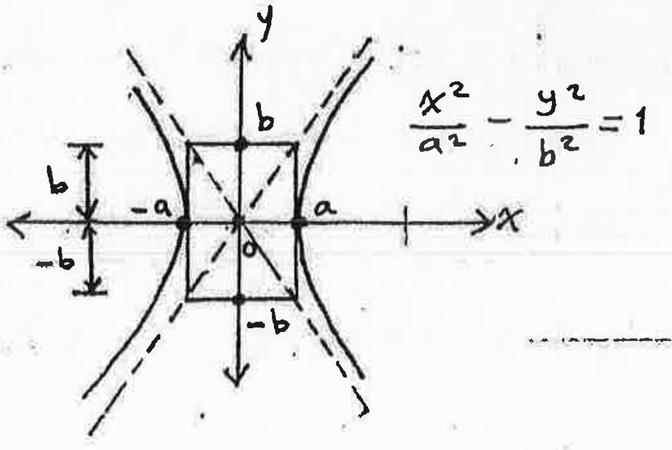
For hyperbola centered at the origin with foci on the x-axis.

and $$\boxed{\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1}$$

For hyperbola centered at the origin with foci on the y-axis.

The distance b obtained from $\boxed{c = \sqrt{a^2 + b^2}}$

The figure below to explain the distance b.



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Note : a represents the distance on focal axis.

Examples about conic sections:

Ex.1: Write the equation for the circle with center (3,-4) and radius 2 units.

sol. $(x-h)^2+(y-k)^2=a^2$

$(x-3)^2+(y-(-4))^2=(2)^2 \Rightarrow (x-3)^2+(y+4)^2=4$

Ex.2: Write the equation for the parabola with focus (0,-3) and directrix $y=3$.

sol. $y=\dfrac{x^2}{4p}=\dfrac{x^2}{4(-3)}=\dfrac{x^2}{-12} \Rightarrow y=-\dfrac{1}{12}x^2$

Ex.3: For the hyperbola $\dfrac{x^2}{4}-\dfrac{y^2}{5}=1$, find the coordinates for the foci and vertices.

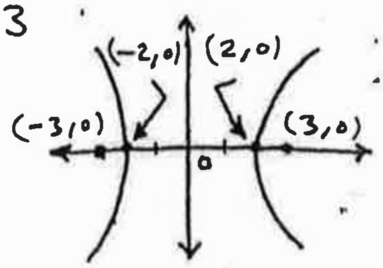
sol. $\dfrac{x^2}{4}-\dfrac{y^2}{5}=1 \Longleftrightarrow \dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$

$\Rightarrow a^2=4$ and $b^2=5 \Rightarrow a=\mp 2$ and $b=\mp\sqrt{5}$

$\Rightarrow c=\sqrt{a^2+b^2}=\sqrt{4+5}=\sqrt{9}=3$

$\Rightarrow$ foci are (3,0) and (-3,0).

vertices are (2,0) and (-2,0).



Ex.4: For the ellipse $\dfrac{x^2}{9}+\dfrac{y^2}{16}=1$, find the coordinates for the foci and vertices.

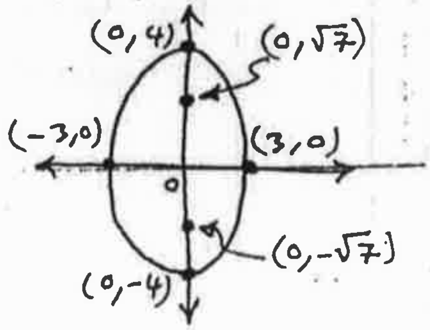
sol. $\dfrac{x^2}{9}+\dfrac{y^2}{16}=1 \Longleftrightarrow \dfrac{x^2}{b^2}+\dfrac{y^2}{a^2}=1$

$\Rightarrow a^2=16$ and $b^2=9 \Rightarrow a=\mp 4$ and $b=\mp 3$

$\Rightarrow c=\sqrt{a^2-b^2}=\sqrt{16-9}=\sqrt{7}$

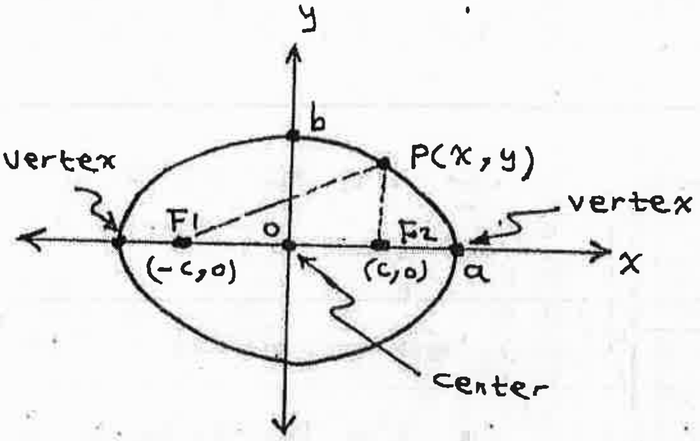
$\Rightarrow$ foci are $(0,\sqrt{7})$ and $(0,-\sqrt{7})$

vertices are (0,4) and (0,-4)



④ Ellipse: An ellipse is the set of points in a plane whose distance from two fixed points have a constant sum. The two fixed points are the foci of the ellipse.

$$\overline{PF_1}+\overline{PF_2}=2a$$



This equation leads to the standard equations for ellipse:

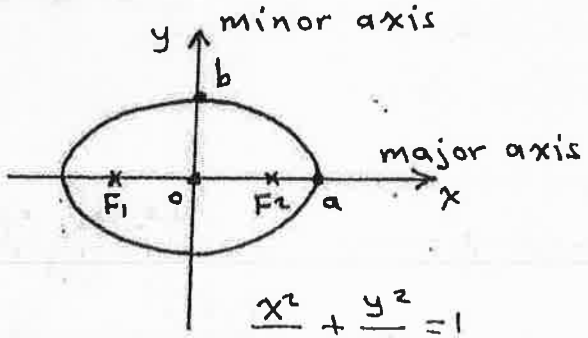
$$\Rightarrow \boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$$ equation for ellipse centered at the origin with foci on the x-axis.

and $$\boxed{\frac{x^2}{b^2}+\frac{y^2}{a^2}=1}$$ equation for ellipse centered at the origin with foci on the y-axis.

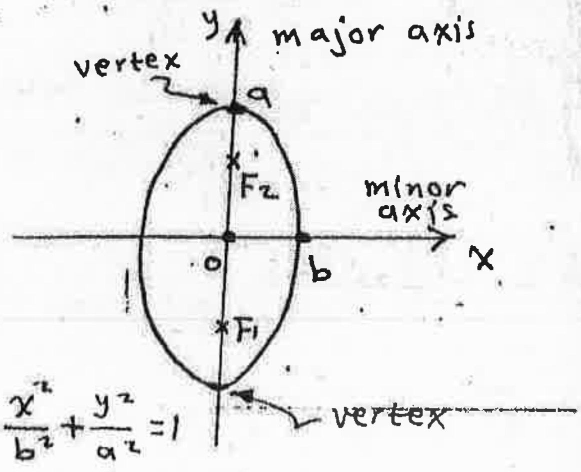
The distance b obtained from $\boxed{c=\sqrt{a^2-b^2}}$

a represents the semimajor axis.

b represents the semiminor axis.



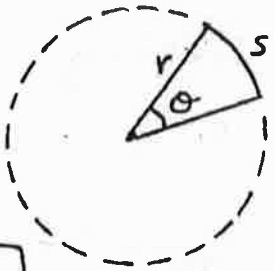
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$$

1.5 : A Review of Trigonometric Functions :

Radian Measure :



$s = \text{arc length} \neq r\theta \Rightarrow \boxed{\theta = \frac{s}{r}}$ —— ①

For semicircle , $\theta = 180^\circ$, $s = \frac{2\pi r}{2} = \pi r$

substitute $s = \pi r$. and $\theta = 180^\circ$ into eq. ①:

$\Longrightarrow \quad 180^\circ = \frac{\pi r}{r} \quad \Longrightarrow \quad \boxed{180^\circ = \pi \text{ radians}}$ —— ②

divide both sides of eq. ② by 180° :

$\Rightarrow \boxed{1 \text{ degree} = \frac{\pi}{180} \text{ radians}}$

divide both sides of eq. ② by π :

$\Rightarrow \boxed{1 \text{ radians} = \frac{180}{\pi} \text{ degrees}} \quad \approx 57.295^\circ$ *

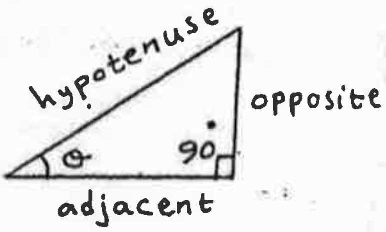
Examples : $45^\circ = 45 * \frac{\pi}{180} = \frac{\pi}{4}$ radians .

$60^\circ = 60 * \frac{\pi}{180} = \frac{\pi}{3}$ rad .

$\frac{\pi}{6}$ rad. $= \frac{\pi}{6} * \frac{180}{\pi} = 30^\circ$.

$\frac{\pi}{2}$ rad. $= \frac{\pi}{2} * \frac{180}{\pi} = 90^\circ$.

The Six Basic Trigonometric Functions :



① Sine : $\sin\theta = \frac{\text{opp.}}{\text{hyp.}}$

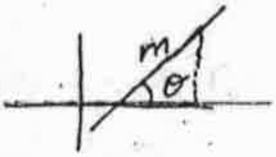
② Cosine : $\cos\theta = \frac{\text{adj.}}{\text{hyp.}}$

③ Tangent : $\tan\theta = \frac{\text{opp.}}{\text{adj.}} = \frac{\sin\theta}{\cos\theta}$

④ Cosecant : $\csc\theta = \frac{\text{hyp.}}{\text{opp.}} = \frac{1}{\sin\theta}$

⑤ Secant : $\sec\theta = \frac{\text{hyp.}}{\text{adj.}} = \frac{1}{\cos\theta}$

⑥ Cotangent : $\cot\theta = \frac{\text{adj.}}{\text{opp.}} = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$

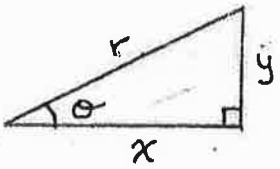


Note : slope of the line = tangent of the inclination angle θ

or $\boxed{m = \tan\theta}$

Identities:

① From Pythogorean theorem:



$$x^2+y^2=r^2$$

By dividing both sides by r^2:

$$\Rightarrow \frac{x^2}{r^2}+\frac{y^2}{r^2}=1$$

$$\Rightarrow \boxed{\cos^2\theta+\sin^2\theta=1}$$

② By dividing this equation by $\sin^2\theta \Rightarrow \boxed{1+\cot^2\theta=\csc^2\theta}$

③ By dividing by $\cos^2\theta \Rightarrow \boxed{1+\tan^2\theta=\sec^2\theta}$

④ $\boxed{\cos(A\mp B)=\cos A\cos B\pm\sin A\sin B}$ ------(a)

$\boxed{\sin(A\mp B)=\sin A\cos B\mp\cos A\sin B}$ ------(b)

⑤ From eq. (a) : $A=B=\theta \Rightarrow \boxed{\cos 2\theta=\cos^2\theta-\sin^2\theta}$

or $\boxed{\cos 2\theta=1-2\sin^2\theta}$

or $\boxed{\cos 2\theta=2\cos^2\theta-1}$

⑥ From eq.(b) : $A=B=\theta \Rightarrow \boxed{\sin 2\theta=2\sin\theta\cos\theta}$

⑦ $$\tan(A\mp B)=\frac{\sin(A\mp B)}{\cos(A\mp B)}$$

$$\Rightarrow \boxed{\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}}$$ ---------(c):

and $$\boxed{\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}}$$

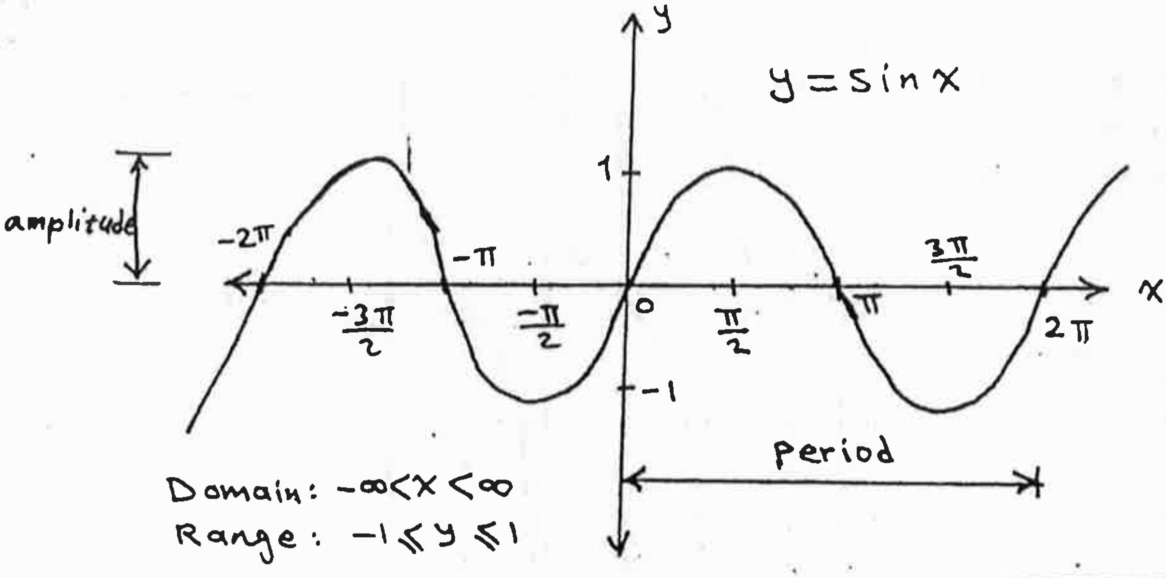
⑧ From eq.(C) ; $A=B=\theta \Rightarrow \boxed{\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}}$

⑨ From the equations $\sin^2\theta+\cos^2\theta=1$ and $\cos 2\theta=\cos^2\theta-\sin^2\theta$:

Add two equations $\Rightarrow \boxed{\cos^2\theta=\frac{1+\cos 2\theta}{2}}$

subtract two equations $\Rightarrow \boxed{\sin^2\theta=\frac{1-\cos 2\theta}{2}}$

⑩ From the graph of the sine function:



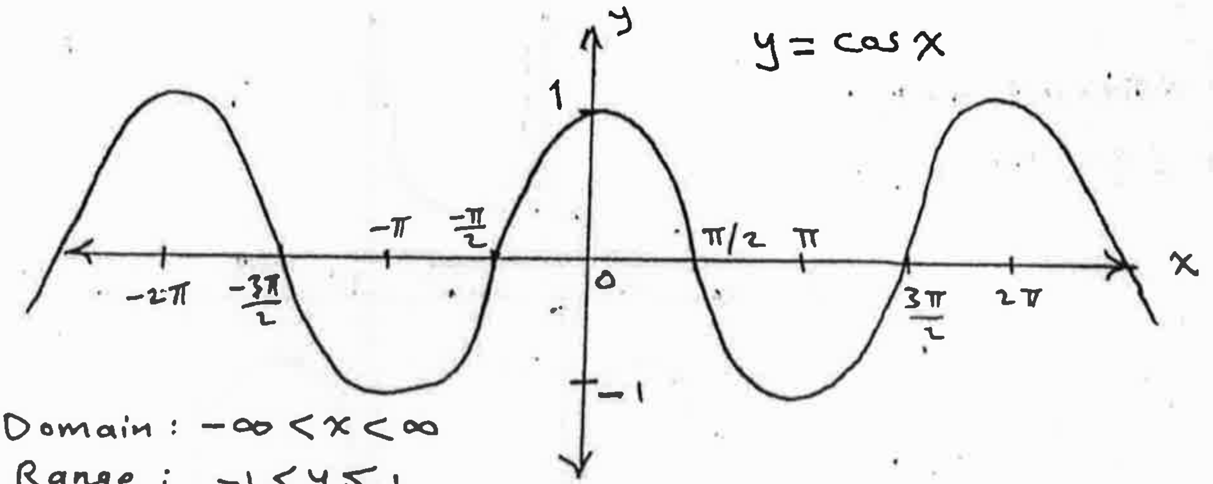
Domain: $-\infty < x < \infty$
Range: $-1 \leqslant y \leqslant 1$

$y = \sin x$ is odd function $\Longrightarrow$ $\boxed{\sin(-x) = -\sin x}$

Also from the graph: $\boxed{\sin x = \sin(x+2\pi) = \sin(x+4\pi) \text{ ---- etc}}$

and $\boxed{\sin x = \sin(x-2\pi) = \sin(x-4\pi) \text{ ---- etc}}$

⑪ From the graph of cosine function:



Domain: $-\infty < x < \infty$
Range: $-1 \leqslant y \leqslant 1$

$y = \cos x$ is even function $\Longrightarrow$ $\boxed{\cos(-x) = \cos x}$

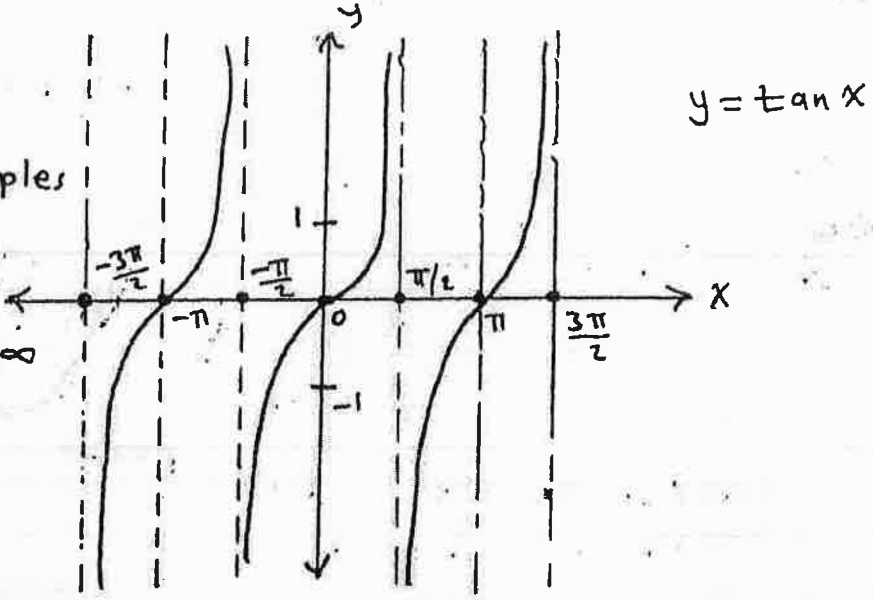
Also from the graph: $\boxed{\cos x = \cos(x+2\pi) = \cos(x+4\pi) \text{ ---- etc}}$

$\boxed{\cos x = \cos(x-2\pi) = \cos(x-4\pi) \text{ ---- etc}}$

⑫ From the graph of tangent function:

Domain: All real numbers except odd integer multiples of $\frac{\pi}{2}$.

Range: $-\infty < y < \infty$



$y = \tan x$ is odd function $\Longrightarrow$ $\boxed{\tan(-x) = -\tan x}$

Also from the graph $\Longrightarrow$ $\boxed{\tan x = \tan(x+\pi) = \tan(x+2\pi) \cdots \text{etc}}$

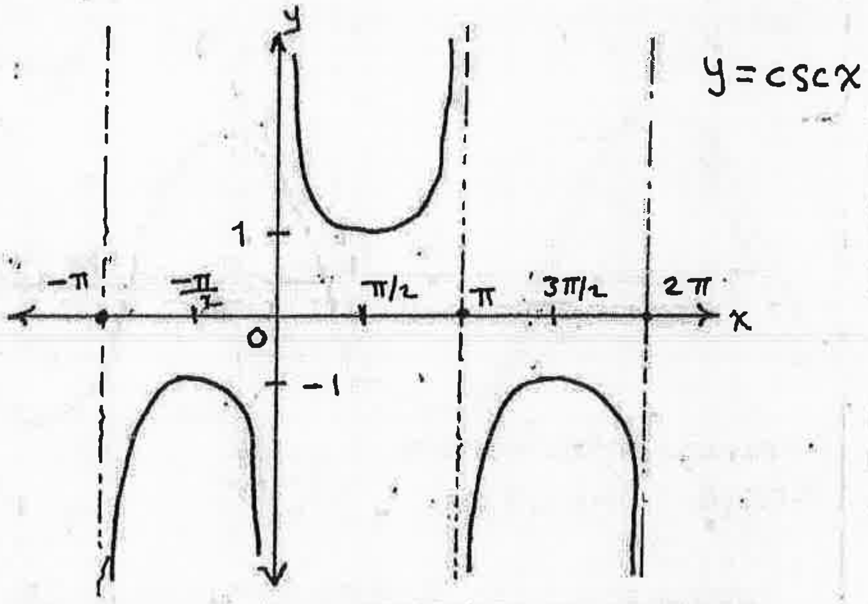
and $\boxed{\tan x = \tan(x-\pi) = \tan(x-2\pi) \cdots \text{etc}}$

⑬ From the graph of cosecant function:

Domain: $x \neq 0, \mp\pi, \mp 2\pi, \cdots$

Range : $y \leq -1$ and $y \geq 1$

*



$y = \csc x$ is odd function $\Longrightarrow$ $\boxed{\csc(-x) = -\csc x}$

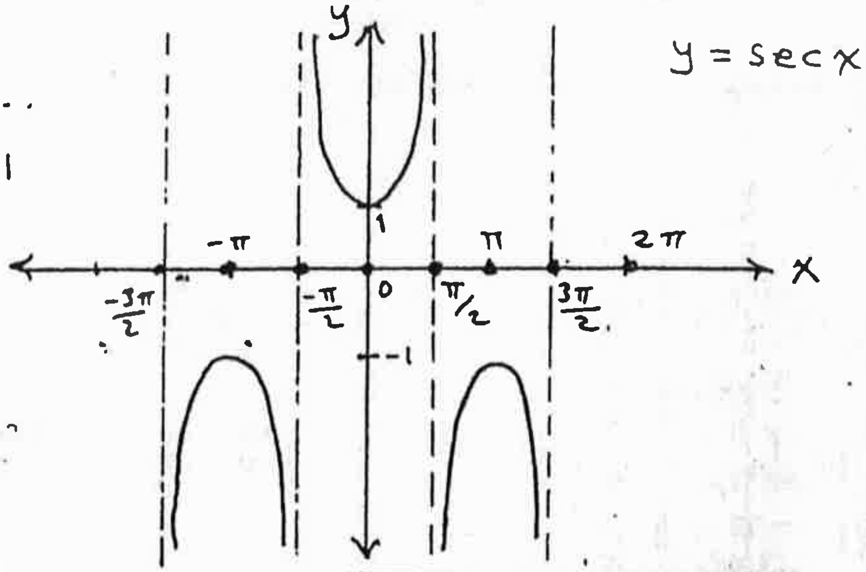
Also from the graph $\Longrightarrow$ $\boxed{\csc x = \csc(x+2\pi) = \csc(x+4\pi) \cdots \text{etc}}$

and $\boxed{\csc x = \csc(x-2\pi) = \csc(x-4\pi) \cdots \text{etc}}$

⑭ From the graph of secant function:

Domain: $x \neq \mp\frac{\pi}{2}, \mp\frac{3\pi}{2}, \ldots$

Range: $y \leqslant -1$ and; $y \geqslant 1$



$y = \sec x$ is even function $\Longrightarrow$ $\boxed{\sec(-x) = \sec(x)}$

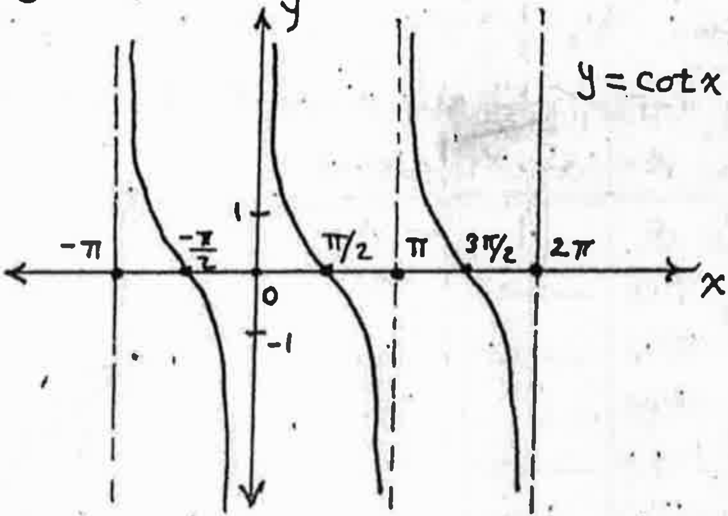
Also from the graph $\Rightarrow$ $\boxed{\sec x = \sec(x+2\pi) = \sec(x+4\pi) \ldots \text{etc}}$

and $\boxed{\sec x = \sec(x-2\pi) = \sec(x-4\pi) \ldots \text{etc}}$

⑮ From the graph of cotangent function:

Domain: $x \neq 0, \mp\pi, \mp 2\pi$

Range : $-\infty < y < \infty$



$y = \cot x$ is odd function $\Rightarrow$ $\boxed{\cot(-x) = -\cot x}$

Also from the graph $\Rightarrow$ $\boxed{\cot x = \cot(x+\pi) = \cot(x+2\pi) \ldots \text{etc}}$

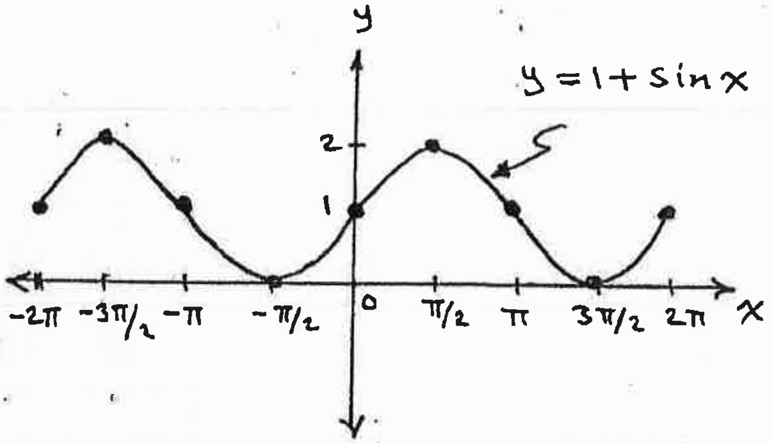
and $\boxed{\cot x = \cot(x-\pi) = \cot(x-2\pi) \ldots \text{etc}}$

Exercises 1.5 / P.54 :

(31) Graph the function $y = 1 + \sin x$, $-2\pi \leq x \leq 2\pi$.

Sol.

x	y
0	1
$\pi/2$	2
π	1
$3\pi/2$	0
2π	1
$-\pi/2$	0
$-\pi$	1
$-3\pi/2$	2
-2π	1

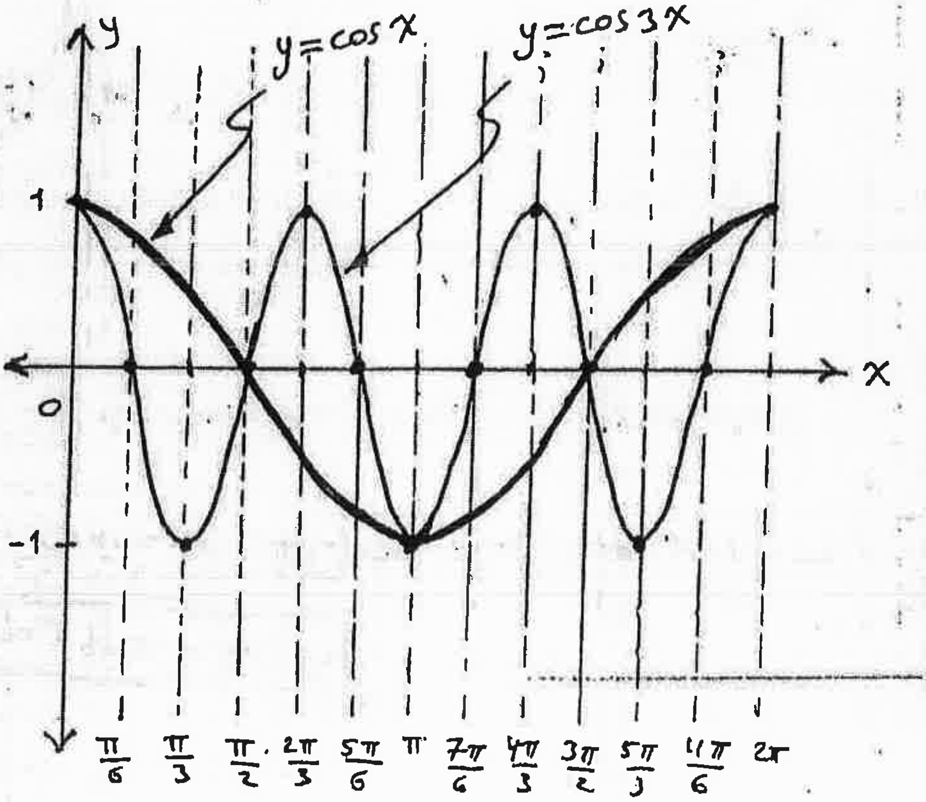


(37) Sketch the graphs of $y = \cos x$ and $y = \cos 3x$, $0 \leq x \leq 2\pi$ together.

Sol. The divisions of $y = \cos x$ is $\frac{\pi}{2}$.

For the divisions of $y = \cos 3x$: $3x = \frac{\pi}{2} \Rightarrow x = \frac{\pi}{6}$

x	$\cos x$	$\cos 3x$
0	1	1
$\pi/6$	—	0
$\pi/3$	—	-1
$\pi/2$	0	0
$2\pi/3$	—	1
$5\pi/6$	—	0
π	-1	-1
$7\pi/6$	—	0
$4\pi/3$	—	1
$3\pi/2$	0	0
$5\pi/3$	—	-1
$11\pi/6$	—	0
2π	1	1



1.6 : Absolute Value (or Magnitude): ~~and Target Values~~

Properties of inequalities:

① $a < b \Rightarrow a + c < b + c$

② $a < b \Rightarrow a - c < b - c$

③ $a < b$ and $c > 0 \Rightarrow ac < bc$

④ $a < b$ and $c > 0 \Rightarrow \frac{a}{c} < \frac{b}{c}$

⑤ $a < b \Rightarrow -b < -a$ [Multiplying both sides with -1 reverses the inequality]

⑥ $a < b \Rightarrow \frac{1}{b} < \frac{1}{a}$ [Taking reciprocal reverses the inequality

⑦ $a < b \Rightarrow \sqrt{a} < \sqrt{b}$

⑧ $|a| < b \Rightarrow -b < a < b$. and $|a| \leqslant b \Rightarrow -b \leqslant a \leqslant b$

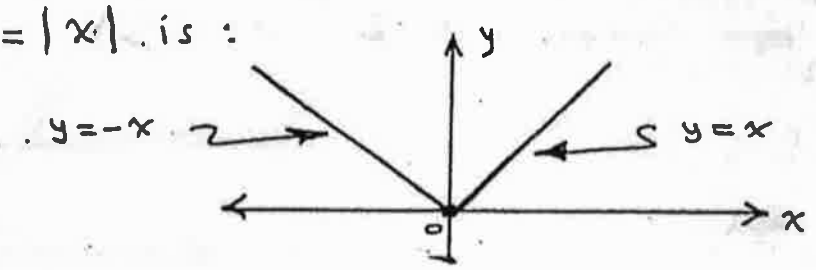
⑨ $|a| > b \Rightarrow a > b$ and $a < -b$. and $a \geqslant b \Rightarrow a \geqslant b$ and $a \leqslant -b$

Absolute value:

The absolute value of a number x denoted by $|x|$, is defined as:

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

The graph of $y = |x|$ is :



Ex.1 Solve the equation $|2x-3| = 7$.

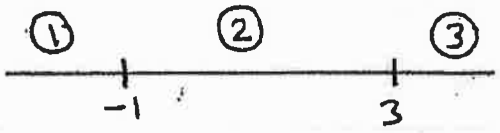
sol. $2x - 3 = 7 \Rightarrow x = 5$

and $-(2x-3) = 7 \Rightarrow -2x + 3 = 7 \Rightarrow x = -2$

Note : In graphing absolute values, divide the x-axis into intervals in which absolute values = 0 , then find the function for each interval.

Ex.2: Graph the function " $y = |x+1| + |x-3|$

sol. $x+1=0 \Rightarrow x=-1$ and $x-3=0 \Rightarrow x=3$.



① For $x<-1$: take -2

$(x+1)$ is $-ve$ and $(x-3)$ is $-ve$

$\Rightarrow -(x+1)-(x-3) = -x-1-x+3$

$= \boxed{2-2x}$

② For $-1<x<3$: take $x=0$

$(x+1)$ is $+ve$ and $(x-3)$ is $-ve$

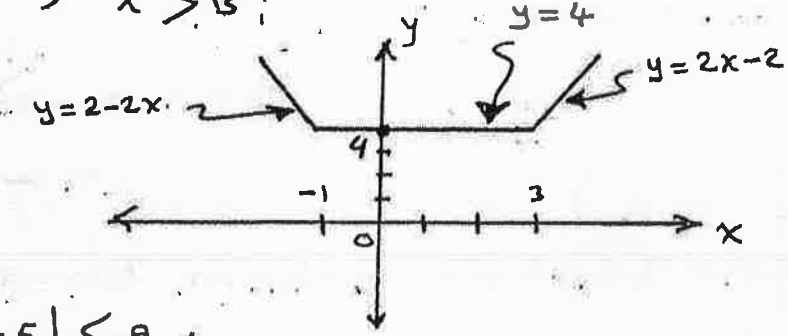
$\Rightarrow (x+1)-(x-3) = x+1-x+3 = \boxed{4}$

③ For $x>3$: take $x=4$

$(x+1)$ is $+ve$ and $(x-3)$ is $+ve$.

$\Rightarrow (x+1)+(x-3) = x+1+x-3 = \boxed{2x-2}$

$$\Rightarrow \quad y = \begin{cases} 2-2x & , \ x<-1 \\ 4 & , \ -1 \leqslant x \leqslant 3 \\ 2x-2 & , \ x>3 ; \end{cases}$$



Ex.3 : Find x in $|x-5|<9$:

sol. $|a|<b \Rightarrow -b<a<b$

$a = x-5 \quad , \ b=9$

$\Rightarrow \quad -9<x-5<9$

For $x-5>-9 \Rightarrow x>-4$

For $x-5<9 \Rightarrow x<14$

$\Rightarrow \quad -4<x<14$

x.4 Find x in $\left|5-\frac{2}{x}\right| \leqslant 1$.

Sol. $|a| \leqslant b \Longrightarrow -b \leqslant a \leqslant b$.

$a = 5-\frac{2}{x}$ and $b = 1$.

$\Rightarrow \quad -1 \leqslant 5-\frac{2}{x} \leqslant 1$

For $5-\frac{2}{x} \geqslant -1 \Longrightarrow \frac{-2}{x} \geqslant -6$

$[* -1] \Longrightarrow \frac{2}{x} \leqslant 6 \Longrightarrow \frac{1}{x} \leqslant 3$

taking reciprocal $\Longrightarrow x \geqslant \frac{1}{3}$

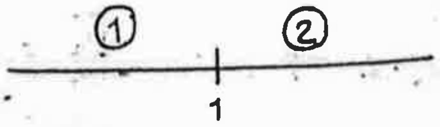
For $5-\frac{2}{x} \leqslant 1 \Longrightarrow -\frac{2}{x} \leqslant -4 \Longrightarrow \frac{2}{x} \geqslant 4 \Rightarrow \frac{1}{x} \geqslant 2$

$\Longrightarrow x \leqslant \frac{1}{2}$

$\Longrightarrow \quad \frac{1}{3} \leqslant x \leqslant \frac{1}{2}$

Exercises 1.6 / P. 63 :

(52) Graph $y = \frac{|x-1|}{x-1}$



Sol. $x-1=0 \Longrightarrow x=1$

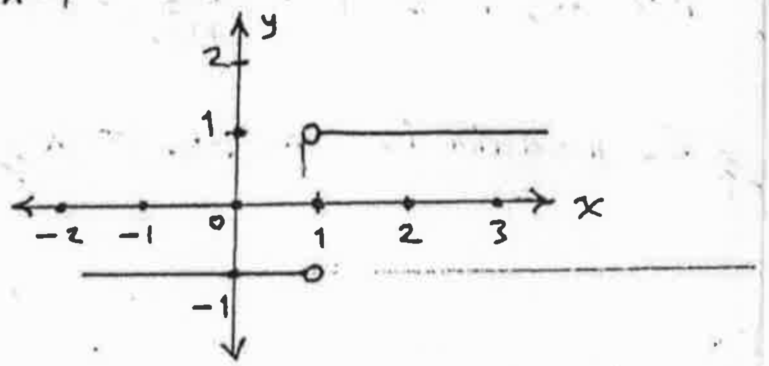
For $x<1$: take $x=0$

$\Rightarrow (x-1)$ is $-ve \Rightarrow y = \frac{-(x-1)}{x-1} = -1$

For $x>1$: take $x=2$

$\Rightarrow (x-1)$ is $+ve \Longrightarrow y = \frac{+(x-1)}{x-1} = 1$

$$\Rightarrow y = \begin{cases} -1, & x<1 \\ 1, & x>1 \end{cases}$$



Supplementary problems :

① Find the domain for $y=\sqrt{x^2-9}$

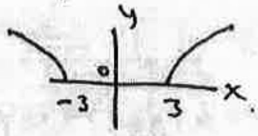
Sol. $x^2-9=0 \Rightarrow x^2=9 \Rightarrow x=3$ and $x=-3$

For $x<-3 \Rightarrow y=+ve$ (o.k.)

For $x=-3 \Rightarrow y=0$ (o.k.)

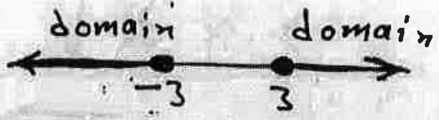
For $-3<x<3 \Rightarrow y=\sqrt{-}$ (not o.k.)

For $x>3 \Rightarrow y=+ve$ (o.k.)



$\Rightarrow$ The domain is all real numbers except $-3<x<3$

or: The domain is $x \geqslant 3$ & $x \leqslant -3$



② Find the domain for $y=\sqrt{\frac{x-1}{x+2}}$

Sol. $x-1=0 \Rightarrow x=1$ & $x+2=0 \Rightarrow x=-2$

For $x<-2 \Rightarrow y=\sqrt{\frac{-ve}{-ve}}=+ve$ (o.k.)

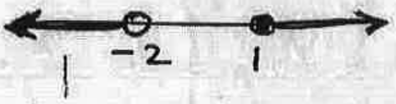
For $x=-2 \Rightarrow y=\sqrt{\frac{-ve}{0}}$ (not o.k.)

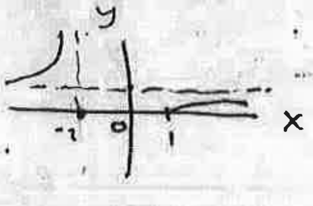
For $-2<x<1 \Rightarrow y=\sqrt{\frac{-ve}{+ve}}=\sqrt{-ve}$ (not o.k.)

For $x=1 \Rightarrow y=\sqrt{\frac{0}{+ve}}=\sqrt{0}=0$ (o.k.)

For $x>1 \Rightarrow y=\sqrt{\frac{+ve}{+ve}}=\sqrt{+ve}$ (o.k.)

$\Rightarrow$ The domain is $x<-2$ & $x \geqslant 1$





CHAPTER TWO

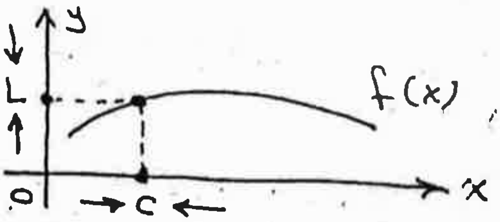
LIMITS AND CONTINUITY

2.1 : Limits :

When the values of a function f(x) approach the value L as x approaches C, we say that f(x) has limit L as x approaches C

Or : $\lim_{x \to c} f(x) = L$.

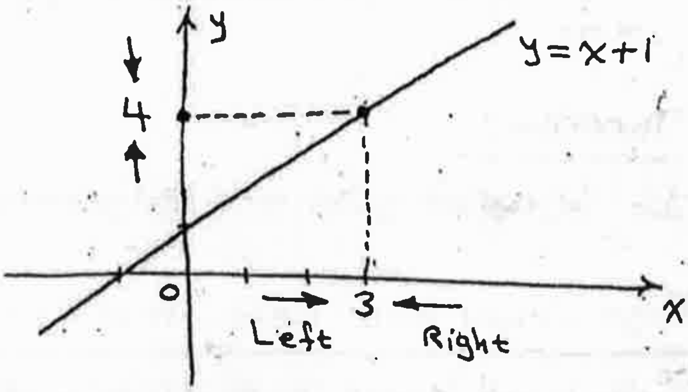
Limit is used to describe the value that a function approaches as the input approaches some value.



Ex. Find the limit of the function $f(x) = x+1$ as x approaches 3

Sol. $\lim_{x \to c} f(x) = \lim_{x \to 3}(x+1)$

$= 3+1 = 4$



Notes :

① For identity function $(f(x)=x)$, $\lim_{x \to c} f(x) = \lim_{x \to c} x = c$.

② For constant function $(f(x)=k)$, $\lim_{x \to c} f(x) = \lim_{x \to c} k = k$.

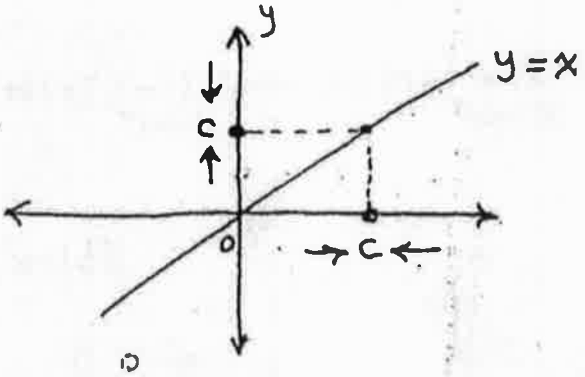


Fig. ①

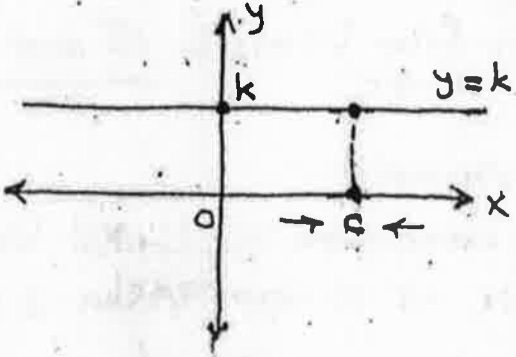


Fig. ②

Theorems :

Theorem 1 : (Properties of limits):

If $\lim\limits_{x\to c} f_1(x) = L_1$ and $\lim\limits_{x\to c} f_2(x) = L_2$, then :

① $\lim\limits_{x\to c} [f_1(x) \mp f_2(x)] = L_1 \mp L_2$ ② $\lim\limits_{x\to c} [f_1(x) . f_2(x)] = L_1 . L_2$

③ $\lim\limits_{x\to c} [f_1(x)/f_2(x)] = L_1/L_2 \; , L_2 \neq 0$ ④ $\lim\limits_{x\to c} [k . f_1(x)] = k . L_1$

k : constant

Theorem 2 :

If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ is any polynomial function then:

$$\lim_{x\to c} f(x) = f(c) = a_n c^n + a_{n-1} c^{n-1} + \dots + a_0$$

Theorem 3 :

If $f(x)$ and $g(x)$ are polynomials, then $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{f(c)}{g(c)} \; , g(c) \neq 0$

Right-hand and Left-hand Limits :

Right-hand limit is the limit of $f(x)$ as x approaches c from the right , or $\lim\limits_{x\to c^+} f(x)$.

Left-hand limit is the limit of $f(x)$ as x approaches c from the left , or $\lim\limits_{x\to c^-} f(x)$.

Note : $\lim\limits_{x\to c} f(x) = L$ *if and only if* $\lim\limits_{x\to c^+} f(x) = L$ and $\lim\limits_{x\to c^-} f(x) = L$.

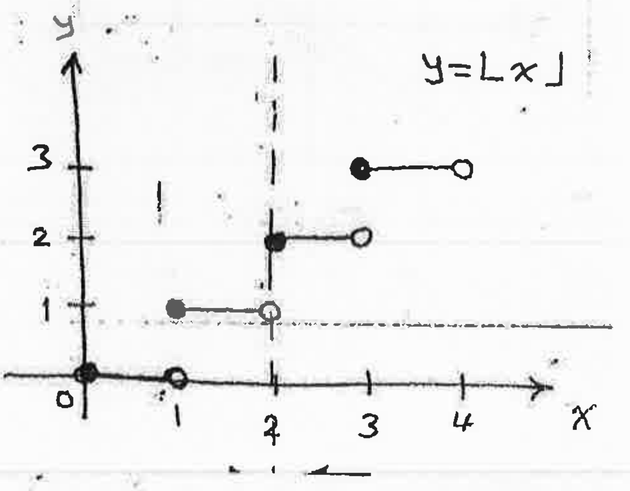
For example :

The function $y = \lfloor x \rfloor$ has no limit as x approaches 2 because:

$\lim\limits_{x\to 2^+} \lfloor x \rfloor = 2$

& $\lim\limits_{x\to 2^-} \lfloor x \rfloor = 1$

$2 \neq 1 \Rightarrow \lim\limits_{x\to 2} \lfloor x \rfloor$ not exist .



Indeterminate Forms :

There are seven indeterminate forms :

$\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \cdot \infty$, $\infty - \infty$, 0^{0} , ∞^{0}, and 1^{∞}.

Exercises 2.1 / P. 86 :

Find the limits :

(22) $\lim\limits_{x \to 5} \frac{4}{x-7} = \frac{4}{5-7} = \frac{4}{-2} = -2$

(36) $\lim\limits_{x \to 2} \frac{x^2-7x+10}{x-2} = \lim\limits_{x \to 2} \frac{\cancel{(x-2)}(x-5)}{\cancel{(x-2)}} = \lim\limits_{x \to 2} x-5 = 2-5 = -3$

(45) Let :

$$f(x) = \begin{cases} 3-x & , x<2 \\ \frac{x}{2}+1 & , x>2 \end{cases}$$

ⓐ Find $\lim\limits_{x \to 2^+} f(x)$ and $\lim\limits_{x \to 2^-} f(x)$.

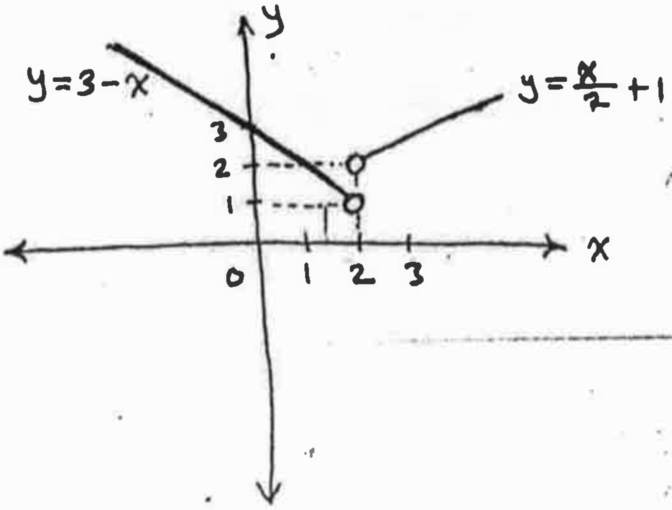
ⓑ Does $\lim\limits_{x \to 2} f(x)$ exist ? Why ?

Sol. ⓐ $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} \left(\frac{x}{2}+1\right) = \frac{2}{2}+1 = 2$.

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (3-x) = 3-2 = 1$

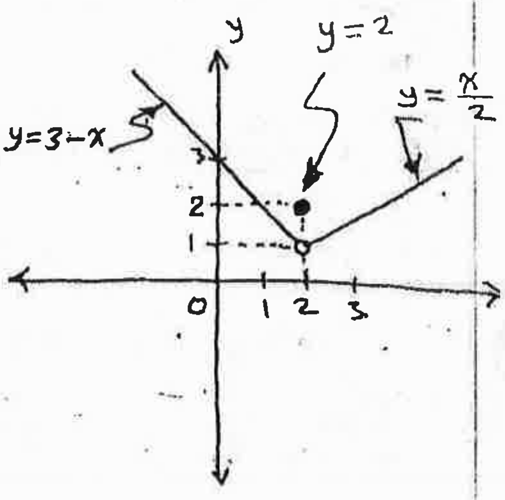
ⓑ $\lim\limits_{x \to 2} f(x)$ does not exist because $\lim\limits_{x \to 2^+} f(x) \neq \lim\limits_{x \to 2^-} f(x)$.

Another method to solve this problem is from the graph :



㊻ Let:

$$f(x)=\begin{cases} 3-x & , x<2 \\ 2 & , x=2 \\ \frac{x}{2} & , x>2 \end{cases}$$



ⓐ Find $\lim_{x\to 2^+} f(x)$ and $\lim_{x\to 2^-} f(x)$.

ⓑ Does $\lim_{x\to 2} f(x)$ exist? Why?

sol. ⓐ $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x}{2} = \frac{2}{2} = 1$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 3-x = 3-2 = 1$

ⓑ $\lim_{x\to 2} f(x)$ exist because $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x)$.

㊿ Let:

$$f(x)=\begin{cases} 1-x^2 & , x\neq 1 \\ 2 & , x=1 \end{cases}$$

ⓐ Find $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$.

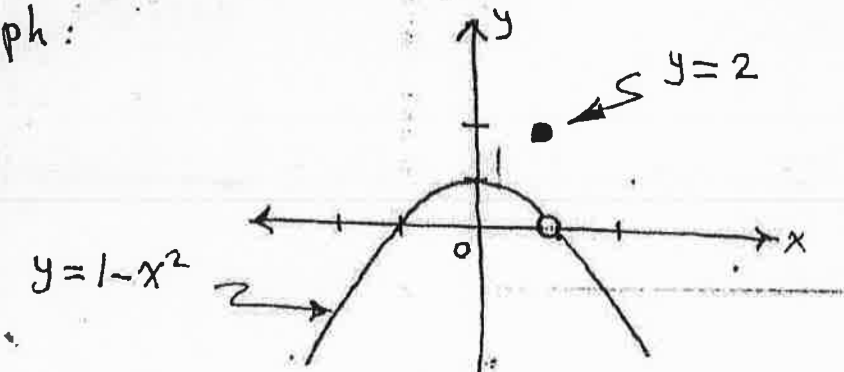
ⓑ Does $\lim_{x\to 1} f(x)$ exist? Why? ⓒ Graph $f(x)$.

sol. ⓐ $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 1-x^2 = 1-(1)^2 = 1-1 = 0$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 1-x^2 = 1-(1)^2 = 1-1 = 0$

ⓑ $\lim_{x\to 1} f(x)$ exist because $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x)$.

ⓒ The graph:



2.2: The Sandwich Theorem and $\sin\theta/\theta$:

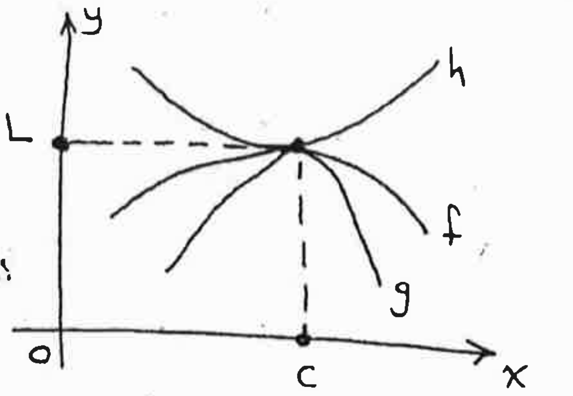
Suppose $g(x) \leqslant f(x) \leqslant h(x)$

The sandwich theorem is:

For all $x \neq c$ and

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$$

then $\lim_{x\to c} f(x) = L$.

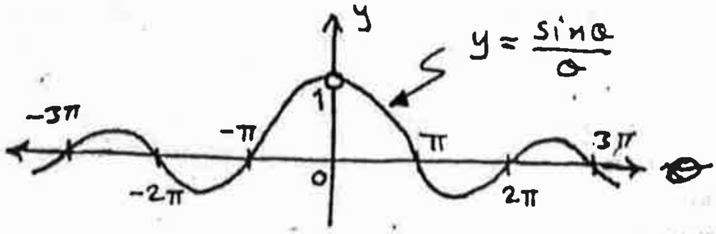


The problem of $\lim_{\theta\to 0} \frac{\sin\theta}{\theta}$:

If we substitute $\theta = 0$ in $\lim_{\theta\to 0} \frac{\sin\theta}{\theta}$, we will get indeterminat form $[\frac{0}{0}]$, but by using sandwich theorem we get:

$$\boxed{\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1}$$

The graph of $y = \frac{\sin\theta}{\theta}$ is:



Ex.1: Find $\lim_{x\to 0} \frac{\sin 3x}{x}$.

sol. let $3x = \theta \Rightarrow x = \frac{\theta}{3}$

$$\Rightarrow \lim_{x\to 0} \frac{\sin 3x}{x} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta/3} = \lim_{\theta\to 0} 3\,\frac{\sin\theta}{\theta} = 3*1 = 3$$

Ex.2: Find $\lim_{x\to 0} \frac{\tan x}{x}$

sol. $$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sin x}{x(\cos x)} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x} = 1*1 = 1$$

Exercises 2.2 / p. 92:

Find the limits:

5) $\lim_{x\to 0^+} \frac{x}{\sin x} = \lim_{x\to 0^+} \frac{x/x}{\sin x/x} = \frac{1}{1} = 1$

1) $\lim_{x\to 0} \frac{\sin x}{2x^2 - x} = \lim_{x\to 0} \frac{\sin x}{x(2x-1)} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{1}{2x-1} = 1 * \frac{1}{-1} = -1$

2) $\lim_{x\to 0} \frac{x + \sin x}{x} = \lim_{x\to 0} \frac{x}{x} + \frac{\sin x}{x} = 1 + 1 = 2$

Supplementary Problems:

① $\lim\limits_{x \to -2} \dfrac{\frac{1}{x}+\frac{1}{2}}{x^3+8}$ Use: $A^3+B^3=(A+B)(A^2-AB+B^2)$

$$= \lim_{x \to -2} \frac{\frac{\cancel{(x+2)}}{2x}}{\cancel{(x+2)}(x^2-2x+4)} = \lim_{x \to -2} \frac{1}{2x(x^2-2x+4)}$$

$$= \frac{1}{2(-2)[-2^2(-2*-2)+4]} = -\frac{1}{48}$$

② $\lim\limits_{x \to 4} \dfrac{3-\sqrt{x+5}}{x-4}$ Multiply by the conjugate of the numerator over itself.

$$= \lim_{x \to 4} \frac{3-\sqrt{x+5}}{x-4} * \frac{3+\sqrt{x+5}}{3+\sqrt{x+5}}$$

Use: $(A-B)(A+B)=A^2-B^2 \Rightarrow \lim\limits_{x \to 4} \dfrac{9-(x+5)}{(x-4)(3+\sqrt{x+5})}$

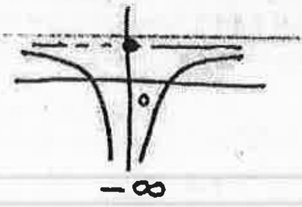
$$= \lim_{x \to 4} \frac{4-x}{(x-4)(3+\sqrt{x+5})} = \lim_{x \to 4} \frac{-\cancel{(x-4)}}{\cancel{(x-4)}(3+\sqrt{x+5})}$$

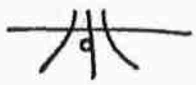
$$= \lim_{x \to 4} \frac{-1}{3+\sqrt{x+5}} = \frac{-1}{3+\sqrt{4+5}} = -\frac{1}{6}$$

③ $\lim\limits_{x \to 0} \dfrac{x^3-7x}{x^3} = \lim\limits_{x \to 0} \dfrac{\cancel{x}(x^2-7)}{\cancel{x}(x^2)} = \lim\limits_{x \to 0} \dfrac{x^2-7}{x^2}$

$= \dfrac{-7}{0^+} = -\infty \Rightarrow$ The limit does not exist.

0^+ means that the value of the denominator x^2 is positive as x approaches zero.





④ $\lim_{x\to0}\frac{x^4+5x-3}{2-\sqrt{x^2+4}} = \frac{-3}{0^-} = \infty \Rightarrow$ the limit does not exist.

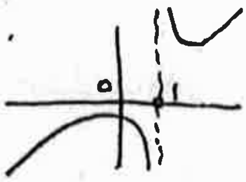
⑤ $\lim_{x\to1}\frac{x^3-1}{(x-1)^2} = \lim_{x\to1}\frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}(x-1)} = \lim_{x\to1}\frac{x^2+x+1}{x-1}$

The numerator approaches 3 and the denominator approaches 0 as x approaches 1.

The quantity in the denominator may be +ve and -ve.

$\Rightarrow$ The answer is neither $+\infty$ nor $-\infty$.

$\Rightarrow$ The limit does not exist.



5) $\lim_{x\to0}\frac{\sin 5x}{3x} = \lim_{x\to0}\frac{5}{5}*\frac{\sin 5x}{3x} = \lim_{x\to0}\frac{5}{3}*\frac{\sin 5x}{5x}$

$= \frac{5}{3}*1 = \frac{5}{3}$

) $\lim_{x\to0}\frac{\cos 2x-1}{\cos x-1}$ Use: $\cos 2x = 2\cos^2 x-1$ $\left[\text{from } \cos^2 x = \frac{1+\cos 2x}{2}\right]$

$= \lim_{x\to0}\frac{(2\cos^2 x-1)-1}{\cos x-1} = \lim_{x\to0}\frac{2\cos^2 x-2}{\cos x-1} = \lim_{x\to0}\frac{2(\cos^2 x-1)}{\cos x-1}$

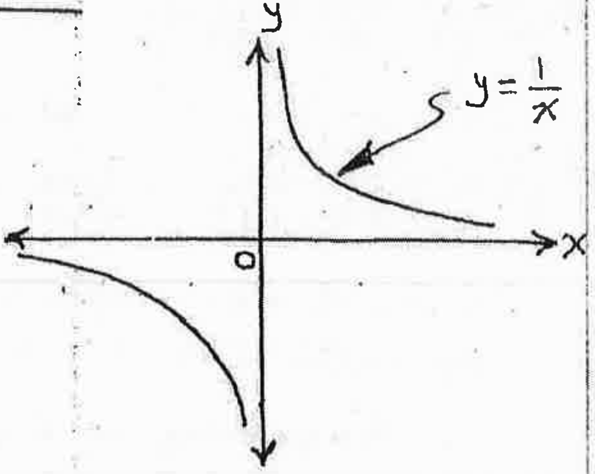
$= \lim_{x\to0}\frac{2\cancel{(\cos x-1)}(\cos x+1)}{\cancel{(\cos x-1)}} = \lim_{x\to0}2(\cos x+1)$

$= 2(\cos 0+1) = 2(1+1) = 4$

3) $\lim_{x\to0}\frac{\tan 2x}{x} = \lim_{x\to0}\frac{\sin 2x}{x(\cos 2x)} = \lim_{x\to0}\frac{2\sin x\cos x}{x\cos 2x}$

$= \lim_{x\to0}\frac{\sin x}{x}\cdot\frac{2\cos x}{\cos 2x} = 1*\frac{2(1)}{1} = 2!$.

2.3 : Limits Involving Infinity :

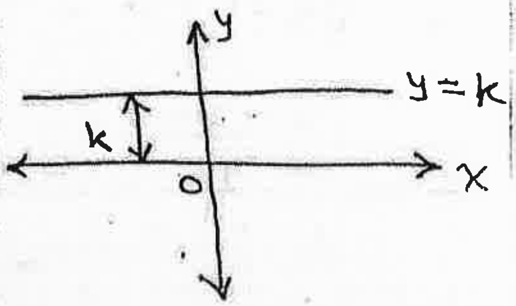


For example, If we have the function $y=\frac{1}{x}$:

$$\lim_{x\to\infty}\frac{1}{x}=0$$

and $$\lim_{x\to-\infty}\frac{1}{x}=0$$

Another example is a constant function $y=k$:



$$\lim_{x\to\infty}k=k$$

and $$\lim_{x\to-\infty}k=k$$

Note : the limit of $\frac{\sin\theta}{\theta}$ as θ approaches $\mp\infty$ is 0.

Ex. Prove that $\lim_{\theta\to\infty}\frac{\sin\theta}{\theta}=0$.

sol. : $-1\leqslant\sin\theta\leqslant1$

$[\div\theta]\Rightarrow\frac{-1}{\theta}\leqslant\frac{\sin\theta}{\theta}\leqslant\frac{1}{\theta}$

Now : $\lim_{\theta\to\infty}\frac{-1}{\theta}=0$ and $\lim_{\theta\to\infty}\frac{1}{\theta}=0$

From Sandwich theorem $\Longrightarrow\lim_{\theta\to\infty}\frac{\sin\theta}{\theta}=0$

Limits of rational functions as x approaches $\mp\infty$:

To solve the limit problems of rational functions as x approaches $\mp\infty$, divide both the numerator and the denominator by the highest power of x in denominator.

Rules: For the rational function $\frac{f(x)}{g(x)}$:

① If degree of $f(x)$ less than degree of $g(x)$ then

$$\lim_{x \to \mp\infty} \frac{f(x)}{g(x)} = 0 .$$

② If degree of $f(x)$ equals degree of $g(x)$ then

$$\lim_{x \to \mp\infty} \frac{f(x)}{g(x)} \text{ is finite.}$$

3) If degree of $f(x)$ greater than degree of $g(x)$ then

$$\lim_{x \to \mp\infty} \quad \text{is infinite.}$$

Examples:

Ex.1: $$\lim_{x \to \infty} \frac{3x+1}{x^2-5} = 0$$ [deg. of numerator less than deg. of denominator].

Ex.2: $$\lim_{x \to \infty} \frac{-x}{7x+3} = \lim_{x \to \infty} \frac{-x/x}{(7x/x)+(3/x)}$$
$$= \lim_{x \to \infty} \frac{-1}{7+\frac{3}{x}} = \frac{-1}{7+0} = \frac{-1}{7}$$

Ex.3: $$\lim_{x \to \infty} \frac{-4x^3+7x}{2x^2-3x-10} = \lim_{x \to \infty} \frac{(-4x^3/x^2)+(7x/x^2)}{(2x^2/x^2)-(3x/x^2)-(10/x^2)}$$
$$= \lim_{x \to \infty} \frac{-4x+(7/x)}{2-(3/x)-(10/x^2)} = \frac{-\infty+0}{2-0-0} = \frac{-\infty}{2} = -\infty$$

Exercises 2.3/P.100:

⑦ Find the limits $\lim_{x\to\infty}$ and $\lim_{x\to-\infty}$ for : $f(x)=\frac{3x^2-6x}{4x-8}$

sol. $\lim_{x\to\infty}\frac{3x^2-6x}{4x-8}=\lim_{x\to\infty}\frac{(3x^2/x)-(6x/x)}{(4x/x)-(8/x)}$

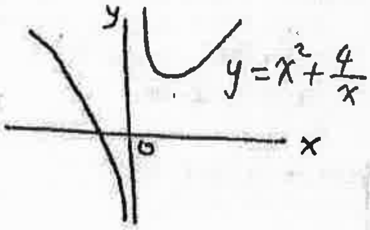
$=\lim_{x\to\infty}\frac{3x-\frac{6x}{x}}{4-\frac{8}{x}}=\frac{\infty-6}{4-0}=\frac{\infty}{4}=\infty$

and $\lim_{x\to-\infty}\frac{3x-\frac{6x}{x}}{4-\frac{8}{x}}=\frac{-\infty-6}{4+0}=\frac{-\infty}{4}=-\infty$

(40) Find $\lim_{x\to0^+}(x^2+\frac{4}{x})$ & $\lim_{x\to0^-}(x^2+\frac{4}{x})$:

sol. $\lim_{x\to0^+}x^2+\frac{4}{x}=0+\infty=\infty$

$\lim_{x\to0^-}x^2+\frac{4}{x}=0-\infty=-\infty$



(54) Find $\lim_{x\to\infty}x\sin\frac{1}{x}$:

sol. Let $\frac{1}{x}=\theta \Rightarrow$ as $x\to\infty$ then $\theta\to0$

$\Rightarrow\lim_{x\to\infty}x\sin\frac{1}{x}=\lim_{\theta\to0}\frac{1}{\theta}\sin\theta=\lim_{\theta\to0}\frac{\sin\theta}{\theta}=1$

(56) $\lim_{x\to\infty}\frac{\cos(1/x)}{1+(1/x)}=\frac{\cos0}{1+0}=\frac{1}{1}=1$

Supplementary Problems:

① $\lim\limits_{x\to\infty} 5x^3 - 2x^2 = \lim\limits_{x\to\infty} x^2(5x-2) = \infty \cdot \infty = \infty$

② $\lim\limits_{x\to\infty} x - \sqrt{x^2+7} = \lim\limits_{x\to\infty} x - \sqrt{x^2+7} * \dfrac{x+\sqrt{x^2+7}}{x+\sqrt{x^2+7}}$

$= \lim\limits_{x\to\infty} \dfrac{x^2-(x^2+7)}{x+\sqrt{x^2+7}} = \lim\limits_{x\to\infty} \dfrac{-7}{x+\sqrt{x^2+7}} = \dfrac{-7}{\infty+\infty} = \dfrac{-7}{\infty} = 0$

③ $\lim\limits_{x\to-\infty} (x - \sqrt{x^2+7}) = -\infty - \infty = -\infty$

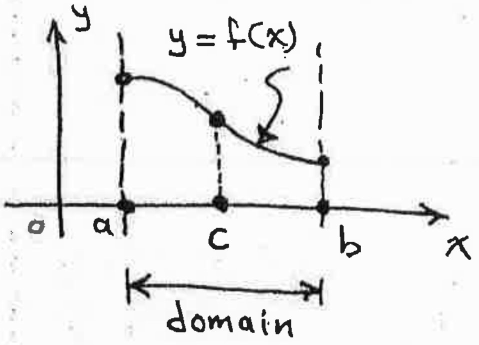
) $\lim\limits_{x\to\infty} \dfrac{x^{3/2}+5}{\sqrt{x^3+4}} = \lim\limits_{x\to\infty} \dfrac{x^{3/2}/x^{3/2} + 5/x^{3/2}}{\sqrt{x^3/x^3 + 4/x^3}}$

$= \lim\limits_{x\to\infty} \dfrac{1+5/x^{3/2}}{\sqrt{1+4/x^3}} = \dfrac{1+0}{\sqrt{1+0}} = \dfrac{1}{\sqrt{1}} = 1$

2.4 : Continuous Functions :

The continuous function is the function that continuous at each point of its domain.

If we have the function $y = f(x)$ with domain in $[a, b]$, and interior point C in the domain, then :



① The function is continuous at C if $\lim_{x \to c} f(x) = f(c)$.

② The function is continuous at a if $\lim_{x \to a^+} f(x) = f(a)$.

③ The function is continuous at b if $\lim_{x \to b^-} f(x) = f(b)$.

Note : If the function is not continuous at a point, we say that this point is the point of discontinuity

Continuity test :

The function $y = f(x)$ is continuous at $x = c$ if and only if all the following three statements are true :

① $f(c)$ exists .

② $\lim_{x \to c} f(x)$ exists.

③ $\lim_{x \to c} f(x) = f(c)$.

Example : Check the continuity of $f(x)$ at $x = 0, 1, 2,$ 3, and 4 .

$$\text{where} \quad f(x) = \begin{cases} 2 - x & , \ 0 \leqslant x < 1 \\ 2 & , \ 1 \leqslant x < 2 \\ 3 & , \ x = 2 \\ 2x - 2 & , \ 2 < x \leqslant 3 \\ 10 - 2x & , \ 3 < x \leqslant 4 \end{cases}$$

Sol. :

at $x = 0$: ① $f(0) = 2 - 0 = 2$

② $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} 2 - x = 2 - 0 = 2$ [end point]

③ $f(0) = \lim\limits_{x \to 0^+} f(x) \Rightarrow$ the function is continuous at $x = 0$.

at $x = 1$: ① $f(1) = 2$

② $\lim\limits_{x \to 1} f(x)$:

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} 2 = 2$

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} 2 - x = 2 - 1 = 1$

$2 \neq 1 \Rightarrow \lim\limits_{x \to 1} f(x)$ does not exist.

$\Rightarrow$ the function is discontinuous at $x = 1$.

at $x = 2$: ① $f(2) = 3$.

② $\lim\limits_{x \to 2} f(x)$:

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} 2x - 2 = 2(2) - 2 = 2$

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} 2 = 2$

$2 = 2 \Rightarrow \lim\limits_{x \to 2} f(x) = 2$

③ $f(2) \neq \lim\limits_{x \to 2} f(x) \Rightarrow$ The function is discontinuous at $x = 2$.

at $x = 3$: ① $f(3) = 2(3) - 2 = 4$ [the portion $2x-2$]

② $\lim\limits_{x \to 3} f(x)$:

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} 10-2x = 10-2(3) = 4$$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} 2x-2 = 2(3)-2 = 4$$

$$4=4 \Rightarrow \lim_{x \to 3} f(x) = 4$$

③ $f(3) = \lim\limits_{x \to 3} f(x)$

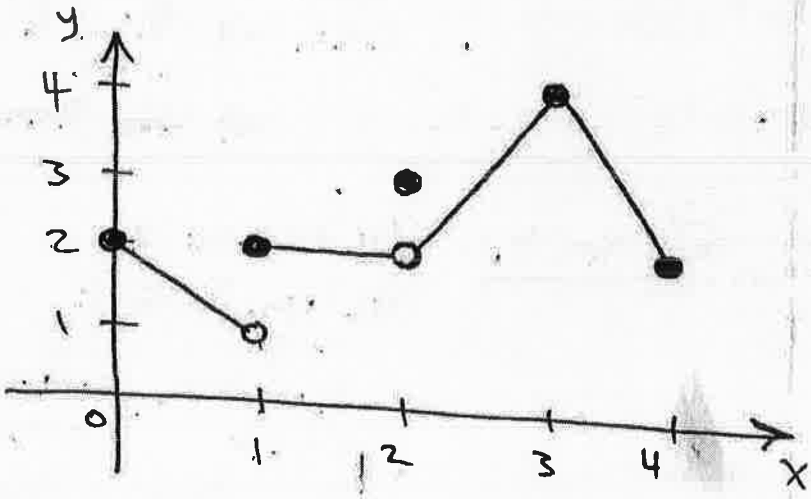
$\Rightarrow$ the function is continuous at $x = 3$.

at $x = 4$: ① $f(4) = 10-2(4) = 2$ [the portion $10-2x$]

[end point] → ② $\lim\limits_{x \to 4^-} f(x) = \lim\limits_{x \to 4^-} 10-2x = 10-2(4) = 2$

③ $f(4) = \lim\limits_{x \to 4^-} f(x) \Rightarrow$ the function is continuous at $x = 4$.

Another method is by graphing the function :



Notes: ① Rational functions are continuous wherever they are defined.

② If f and g are continuous functions at $x=c$, then the following combinations are continuous at $x=c$:

$f+g$, $f-g$, $f\cdot g$, $k\cdot g$, f/g $(g\neq 0)$.

③ If f is continuous at $x=c$ and g is continuous at $f(c)$, then the composite $g\circ f$ is continuous at $x=c$.

Ex. Show that $y=\left|\dfrac{x\cos x}{x^2+2}\right|$ is continuous at every point of its domain.

sol.: $y=g(f(x))$ in which $f(x)=\dfrac{x\cos x}{x^2+2}$

and $g(x)=|x|$.

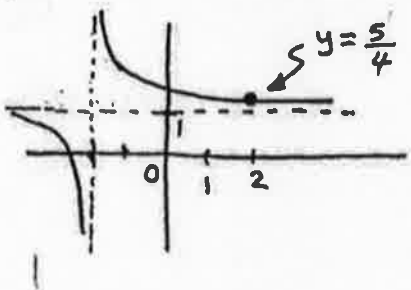
$f(x)$ & $g(x)$ are both continuous $\Rightarrow g\circ f$ is continuous

The continuous extension:

Ex. What is the continuous extension of the function $f(x)=\dfrac{x^2+x-6}{x^2-4}$ to the point $x=2$?

sol. $f(x)=\dfrac{(x+3)\cancel{(x-2)}}{(x+2)\cancel{(x-2)}}=\dfrac{x+3}{x+2}$

$f(2)=\dfrac{2+3}{2+2}=\dfrac{5}{4}$



$$\Rightarrow f(x)=\begin{cases}\dfrac{x^2+x-6}{x^2-4}, & x\neq 2\\ 5/4, & x=2\end{cases}$$

this form is called the continuous extension of the original function to the $x=2$.

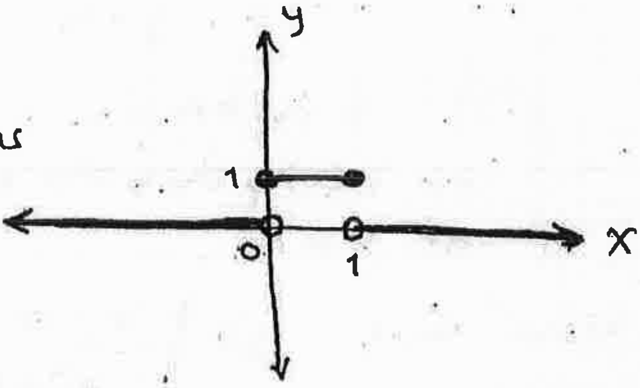
Exercises 2.4 / P.111 :

⑬ At what points is the function f(x) continuous?

$$f(x) = \begin{cases} 0 & , x < 0 \\ 1 & , 0 \leq x \leq 1 \\ 0 & , x > 1 \end{cases}$$

Sol. : from the graph:

this function is continuous at all points except at $x = 0$ and $x = 1$.



⑭ At what points is the function f(x) continuous?

$$f(x) = \begin{cases} 1 & , x < 0 \\ \sqrt{1-x^2} & , 0 \leq x \leq 1 \\ x-1 & , x > 1 \end{cases}$$

Sol. at $x = 0$: ① $f(0) = \sqrt{1-(0)^2} = \sqrt{1} = 1$

② $\lim\limits_{x\to 0} f(x)$: $\lim\limits_{x\to 0^+} f(x) = \sqrt{1-(0)^2} = \sqrt{1} = 1$

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} 1 = 1$

$\Rightarrow \lim\limits_{x\to 0} f(x) = 1$

③ $f(0) = \lim\limits_{x\to 0} f(x) \Rightarrow f(x)$ is continuous at $x=0$.

at $x = 1$: ① $f(1) = 0$

② $\lim\limits_{x\to 1} f(x)$: $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} x - 1 = 1 - 1 = 0$

$\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} \sqrt{1-(1)^2} = \sqrt{0} = 0$

$\Rightarrow \lim\limits_{x\to 1} f(x) = 0$

③ $f(1) = \lim\limits_{x\to 1} f(x) \Rightarrow f(x)$ continuous at $x=1$.

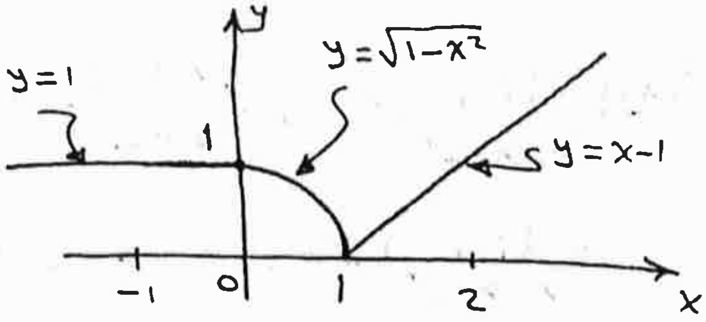
>>>

For $x<0$, $f(x)=1$ is continuous function.

For $x>1$, $f(x)=x-1$ is continuous function.

$\Longrightarrow$ $f(x)$ is continuous at every points.

<u>OR</u>: From the graph:

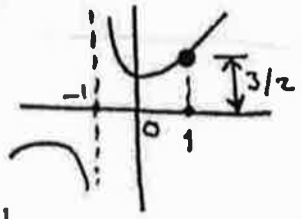


(21) Find the points at which $y=\dfrac{x^3-1}{x^2-1}$ is discontinuous.

<u>sol.</u> $x^2-1=0 \Rightarrow x^2=1 \Longrightarrow \boxed{x=1}$ & $\boxed{x=-1}$

(28) Define $f(1)$ in a way that extends $f(x)=\dfrac{x^3-1}{x^2-1}$ to be continuous.

<u>sol.</u> $f(x)=\dfrac{x^3-1}{x^2-1}=\dfrac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}(x+1)}=\dfrac{x^2+x+1}{x+1}$



$\Rightarrow f(1)=\dfrac{1^2+1+1}{1+1}=\dfrac{3}{2}$

$\Rightarrow$ the extended $f(x)$ is : $f(x)=\begin{cases}\dfrac{x^3-1}{x^2-1} & , x\neq 1\\ 3/2 & , x=1\end{cases}$

(31) What value should be assigned to a to make the function $f(x)=\begin{cases}x^2-1 & , x<3\\ 2ax & , x\geqslant 3\end{cases}$ continuous at $x=3$?

<u>sol.</u> To make $f(x)$ continuous at $x=3$;

$\lim\limits_{x\to 3} f(x)=f(3)$

$\lim\limits_{x\to 3^-} f(x)=\lim\limits_{x\to 3^-} x^2-1=(3)^2-1=8 \Rightarrow \lim\limits_{x\to 3} f(x)=\boxed{8}$

$f(3)=2ax=2*a*3=\boxed{6a} \Longrightarrow 8=6a \Rightarrow a=\dfrac{8}{6}=\dfrac{4}{3}$

Absolute value in the limit problems:

This example clarify the evaluation of the limit of the functions that have absolute values.

Ex.1 $\lim\limits_{x \to -2^+} (x+3)\dfrac{|x+2|}{x+2} = \lim\limits_{x \to -2^+} (x+3)\dfrac{(x+2)}{(x+2)}$

$= \lim\limits_{x \to -2^+} (x+3) = -2+3 = 1$ $\quad [|x+2| = (x+2) \text{ for } x > -2]$

Ex.2: $\lim\limits_{x \to -2^-} (x+3)\dfrac{|x+2|}{x+2}$

$= \lim\limits_{x \to -2^-} (x+3)\dfrac{-(x+2)}{(x+2)}$ $\quad [|x+2| = -(x+2) \text{ for } x < -2]$

$= \lim\limits_{x \to -2^-} (x+3)(-1) = \lim\limits_{x \to -2^-} (-2+3)(-1) = -1$

CHAPTER THREE

DERIVATIVES

Derivative is a function used to measure the rates at which things change.

The derivative can be defined as a limiting values of average changes.

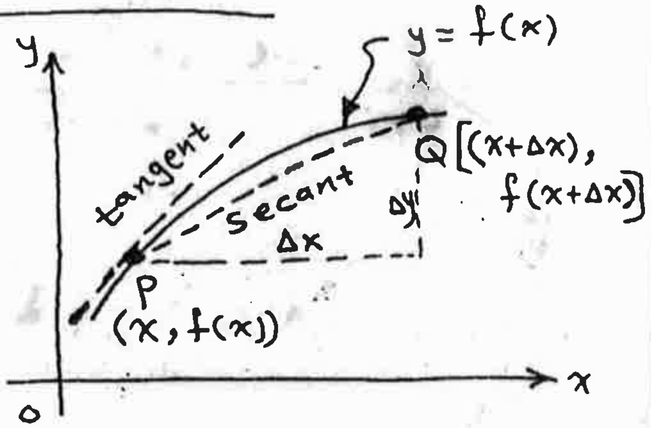
3.1 : Slope, Tangent Line, and Derivative:

If we have the function $y=f(x)$:

From point p to point Q:

slope of secant line =

average rate of change = $\frac{\Delta y}{\Delta x}$



slope of tangent line at p = derivative of f(x) at p.

To find the derivative of the function $y=f(x)$ at point p:

The slope of the tangent at p is the limit of the secant slopes as Q approaches p along the curve.

or Derivative of $f(x) = \lim_{\Delta x\to 0}$ [secant slopes]

but secant slope = $\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$

$$\Rightarrow \boxed{\text{Derivative of } f(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}}$$

The derivative is denoted by : y', $\frac{dy}{dx}$, $f'(x)$, $\frac{df}{dx}$, and $D_x(f)$. And the second derivative is denoted by : y'', $\frac{d^2y}{dx^2}$, $f''(x)$, ...

Ex. If $f(x)=\frac{1}{x}$, find $f'(x)$:

Sol: $f(x)=\frac{1}{x}$, $f(x+\Delta x)=\frac{1}{x+\Delta x}$

$$\Rightarrow f(x+\Delta x)-f(x)=\frac{1}{x+\Delta x}-\frac{1}{x}=\frac{x-(x+\Delta x)}{x(x+\Delta x)}$$

$$=\frac{-\Delta x}{x(x+\Delta x)}$$

$$\Rightarrow \frac{f(x+\Delta x)-f(x)}{\Delta x}=\frac{-1}{x(x+\Delta x)}$$

$$\Rightarrow \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{-1}{x(x+\Delta x)}=\frac{-1}{x(x+0)}=\frac{-1}{x^2}$$

Definitions:

① A function that has a derivative at a point x is said to be differentiable at x.

② A function that has a derivative at every point of its domain is called differentiable.

Theorem: If $f(x)$ has derivative at $x=c$, then $f(x)$ is continuous at $x=c$.

Note: The function has derivative at a point if and only if the right-hand and left-hand derivatives are equal.

Ex. Is the function $y=|x|$ has derivative at $x=0$? Why?

Sol. $$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

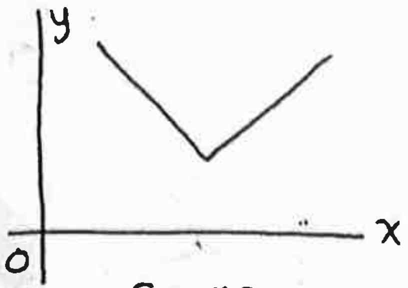
at $x=0 \Rightarrow f'(x)=\lim_{\Delta x\to 0}\frac{|x+\Delta x|-|x|}{\Delta x}=\lim_{\Delta x\to 0}\frac{|0+\Delta x|-|0|}{\Delta x}$

$$=\lim_{\Delta x\to 0}\frac{|\Delta x|}{\Delta x}$$

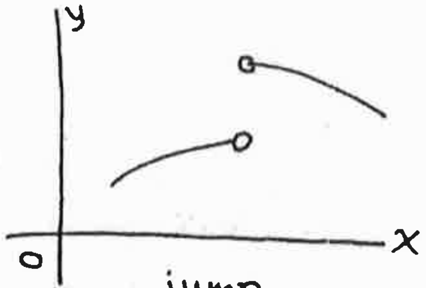
If $\Delta x>0 \Rightarrow f'(x)=\lim_{\Delta x\to 0^+}\frac{\Delta x}{\Delta x}=1$ & if $\Delta x<0 \Rightarrow f'(x)=\lim_{\Delta x\to 0^-}\frac{-\Delta x}{\Delta x}=-1$

$1\neq -1 \Rightarrow y=|x|$ has no derivative at $x=0$.

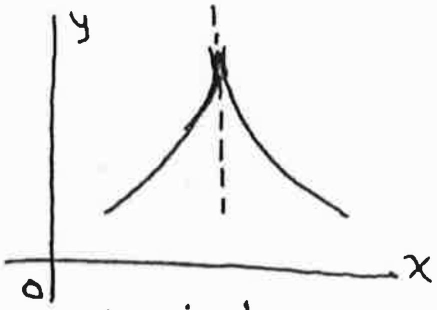
The cases in which the function has not derivative:



corner



jump



vertical tangent

3.2 : Differentiation Rules:

① $\frac{d}{dx} c = 0$, c : constant .

② $\frac{d}{dx} x^n = n x^{n-1}$, n : number.

③ $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$, u and v are differentiable functions.

④ $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

⑤ $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Examples :

① $\frac{d}{dx} 3 = 0$

② $\frac{d}{dx} x^4 = 4x^3$

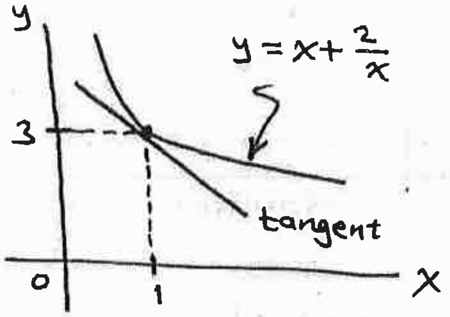
③ $\frac{d}{dx}(x^2+5x) = 2x+5$

④ $\frac{d}{dx}(x^2+1)(x-1) = (x^2+1)(1) + (x-1)(2x)$

$= x^2+1 + 2x^2-2x = 3x^2-2x+1$

⑤ $\frac{d}{dx}\frac{x^2-1}{x^2+1} = \frac{(x^2+1)(2x)-(x^2-1)(2x)}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$

Ex. Find the equation for the tangent to the curve $y = x + \frac{2}{x}$ at the point $(1,3)$.



Sol.: slope of the tangent at a point p. = derivative of the function at p.

We have : $y = x + \frac{2}{x} = x + 2x^{-1}$

$$\Rightarrow \frac{dy}{dx} = 1 - 2x^{-2}$$

$$= 1 - \frac{2}{x^2}$$ [the derivative at any point].

at (1,3) : $\frac{dy}{dx} = 1 - \frac{2}{(1)^2} = 1 - 2 = -1$

$\Rightarrow$ slope of the tangent line $= -1 = m$

$\Rightarrow$ the equation for the tangent is :

$y - y_1 = m(x - x_1)$

$y - 3 = -1(x-1) \Rightarrow y = -x+1+3 \Rightarrow y = 4 - x$

3.3: Velocity and Speed :

① Velocity: is the rate of change in the position of a body.

$v = \frac{ds}{dt}$, s : displacement (change in position).

② Speed : is the absolute value of velocity.

speed = | velocity |

③ Acceleration : is the rate of change in the velocity of a moving body.

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

④ The equation of free fall of a body in a vacuum is :

$s = \frac{1}{2} g t^2$ where s : distance of fall.
t: time of fall
g: gravity acceleration $= 9.8 m/s^2$
$= 32 ft/s^2$

3.4: Derivatives of Trigonometric Functions:

① The derivative of the sine:

$$y = \sin x \implies f(x) = \sin x \quad \text{and} \quad f(x+\Delta x) = \sin(x+\Delta x)$$

$$\implies \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin x(\cos \Delta x - 1) + \cos x \sin \Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \sin x \cdot \lim_{\Delta x \to 0} \frac{\cos \Delta x - 1}{\Delta x} + \lim_{\Delta x \to 0} \cos x \cdot \lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} \to \circledast$$

Now $$\lim_{\Delta x \to 0} \frac{\cos \Delta x - 1}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cos \Delta x - 1}{\Delta x} * \frac{\cos \Delta x + 1}{\cos \Delta x + 1}$$

$$= \lim_{\Delta x \to 0} \frac{\cos^2 \Delta x - 1}{\Delta x(\cos \Delta x + 1)} = \lim_{\Delta x \to 0} \frac{-\sin^2 \Delta x}{\Delta x(\cos \Delta x + 1)}$$

$$= -\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} \cdot \lim_{\Delta x \to 0} \frac{\sin \Delta x}{\cos \Delta x + 1} = -1\left(\frac{0}{1+1}\right) = 0$$

Return $\circledast \implies \dfrac{dy}{dx} = \sin x(0) + \cos x(1) = 0 + \cos x = \cos x$

$$\implies \boxed{\frac{d}{dx} \sin x = \cos x}$$

② $\dfrac{d}{dx} \cos x = -\sin x$

③ $\dfrac{d}{dx} \tan x = \sec^2 x$

④ $\dfrac{d}{dx} \cot x = -\csc^2 x$

⑤ $\dfrac{d}{dx} \sec x = \sec x \tan x$

⑥ $\dfrac{d}{dx} \csc x = -\csc x \cot x$

Examples :

① $y = \tan x \cot x$

$$\frac{dy}{dx} = \tan x\,(-\csc^2 x) + \cot x\,(\sec^2 x)$$

$$= \frac{\cancel{\sin x}}{\cos x}\left(-\frac{1}{\sin^{\cancel{2}} x}\right) + \frac{\cancel{\cos x}}{\sin x}\left(\frac{1}{\cos^{\cancel{2}} x}\right)$$

$$= \frac{-1}{\cos x \sin x} + \frac{1}{\cos x \sin x} = 0$$

② $y = \dfrac{\sin x}{1 + \cos x}$

$$\frac{dy}{dx} = \frac{(1+\cos x)(\cos x) - \sin x\,(-\sin x)}{(1+\cos x)^2} = \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$$

but $\cos^2 x + \sin^2 x = 1 \Rightarrow \dfrac{dy}{dx} = \dfrac{1+\cos x}{(1+\cos x)^2} = \dfrac{1}{1+\cos x}$

③ Find the points on the curve $y = \tan x$, $\frac{-\pi}{2} < x < \frac{\pi}{2}$, where the tangent is parallel to the line $y = 2x$.

sol. slope of the line $y = 2x$ is 2 [from $y = mx + b$]

$\Rightarrow \dfrac{dy}{dx}$ for the curve $y = \tan x$ should be equals 2.

$$\frac{dy}{dx} = \sec^2 x = \frac{1}{\cos^2 x} \Rightarrow \frac{1}{\cos^2 x} = 2$$

$$\Rightarrow \cos^2 x = \frac{1}{2} \Rightarrow \cos x = \mp \frac{1}{\sqrt{2}}$$

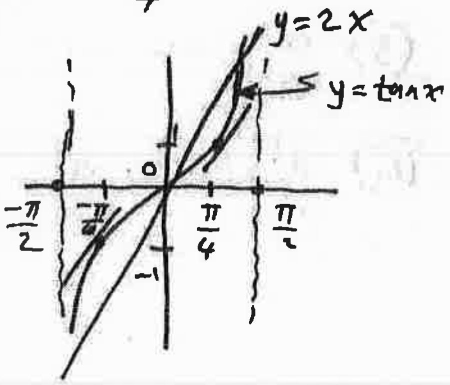
If $\cos x = -\frac{1}{\sqrt{2}} \Rightarrow x$ out of interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.

If $\cos x = \frac{1}{\sqrt{2}} \Rightarrow x = \frac{\pi}{4}$ & $x = -\frac{\pi}{4}$

For $x = \frac{\pi}{4} \Rightarrow y = \tan\frac{\pi}{4} = 1$

For $x = \frac{-\pi}{4} \Rightarrow y = \tan\frac{-\pi}{4} = -1$

$\Rightarrow$ the points are $(\frac{\pi}{4}, 1)$ & $(-\frac{\pi}{4}, -1)$



3.5 : The Chain Rule :

The chain rule is used to differentiate the composite functions.

If $y=f(u)$ and $u=g(x)$, then

$$\boxed{\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}}$$ the chain rule.

$$\begin{tikzcd}
y=f(u) \arrow[r, bend left, "dy/du"] \arrow[rr, bend left=50, "dy/dx"] & u=g(x) \arrow[r, bend left, "du/dx"] & x
\end{tikzcd}$$

Ex.1: Find y' for $y=\sin(x^2+6)$

sol. Let $y=\sin u$ and $u=x^2+6$

$$\Rightarrow \frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\cos u\cdot 2x=\cos(x^2+6)\cdot 2x$$
$$=2x\cos(x^2+6)$$

Ex.2: Find $\frac{dy}{dx}$ for $y=(x^2+3)^4$

sol. $u=x^2+3 \Rightarrow y=(u)^4$

$$\Rightarrow \frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=4u^3\cdot 2x=4(x^2+3)^3\cdot 2x=8x(x^2+3)^3$$

Differentiation Rules for Composite Functions $[u=f(x)]$:

① $\frac{d}{dx}u^n=nu^{n-1}\cdot\frac{du}{dx}$

② $\frac{d}{dx}\sin u=\cos u\cdot\frac{du}{dx}$

③ $\frac{d}{dx}\cos u=-\sin u\cdot\frac{du}{dx}$

④ $\frac{d}{dx}\tan u=\sec^2 u\cdot\frac{du}{dx}$

⑤ $\frac{d}{dx}\cot u=-\csc^2 u\cdot\frac{du}{dx}$

⑥ $\frac{d}{dx}\sec u=\sec u\tan u\cdot\frac{du}{dx}$

⑦ $\frac{d}{dx}\csc u=-\csc u\cot u\cdot\frac{du}{dx}$

Repeated Use for Chain Rule :

If $y=f(u)$, $u=g(v)$, and $v=h(x)$, then :

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}$$

Ex.1: If $y=\tan\sqrt{x}$, find dy/dx .

Sol. $\frac{dy}{dx}=\frac{d}{dx}\tan u=\sec^2 u\cdot\frac{du}{dx}=\sec^2\sqrt{x}\cdot\frac{1}{2\sqrt{x}}$

Ex.2: If $y=\cos(\cos x)$ find dy/dx.

Sol. $\frac{dy}{dx}=-\sin u\cdot\frac{du}{dx}=-\sin(\cos x)\cdot(-\sin x)$

$=\sin x\cdot\sin(\cos x)$

Ex.3: If $y=\sin(1+\tan 2x)$, find dy/dx.

Sol. $\frac{dy}{dx}=\cos(1+\tan 2x)*(0+\sec^2 2x)*2$

$=2\cos(1+\tan 2x)\sec^2 2x$

this is a repeated use where :

$y=f(u)=\sin u$

$u=g(v)=1+\tan v$

and $v=h(x)=2x$

$$\Longrightarrow\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}$$

Ex.4: If $y=\cos^2 3x$, find dy/dx :

Sol. $y=\cos^2 3x=(\cos 3x)^2$

$y=f(u)=u^2$

$u=g(v)=\cos v$

$v=h(x)=3x$

$\Longrightarrow\frac{dy}{dx}=2u\cdot(-\sin v)\cdot 3=2\cos 3x*(-\sin 3x)*3$

$=-6\sin 3x\cos 3x=-3\sin 6x$

3.6: Implicit Differentiation:

The implicit differentiation is used to differentiate the equations that do not have the value of y in terms of x.

Steps for solution:

1- Differentiate both sides of the equation with respect to x.

2- Collect the terms of dy/dx on one side.

3- Factor out dy/dx.

4- Solve for dy/dx.

Ex.1: Find dy/dx for $2y = x^2 + \sin y$.

sol.
$$2\frac{dy}{dx} = 2x + \cos y \frac{dy}{dx}$$
$$2\frac{dy}{dx} - \cos y \frac{dy}{dx} = 2x$$
$$\frac{dy}{dx}(2 - \cos y) = 2x$$
$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x}{2 - \cos y}$$

Ex.2: Find $\frac{d^2y}{dx^2}$ if $2x^3 - 3y^2 = 7$.

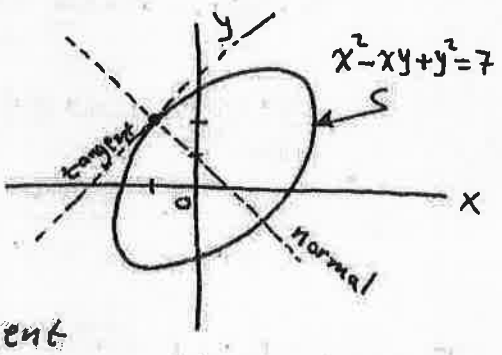
sol.
$$6x^2 - 6y\frac{dy}{dx} = 0 \Rightarrow 6y\frac{dy}{dx} = 6x^2 \Rightarrow \frac{dy}{dx} = \frac{x^2}{y}$$
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{x^2}{y}\right) = \frac{y(2x) - x^2\left(\frac{dy}{dx}\right)}{y^2}$$
$$= \frac{2xy}{y^2} - \frac{x^2\left(\frac{dy}{dx}\right)}{y^2} = \frac{2x}{y} - \frac{x^2}{y^2}\left(\frac{dy}{dx}\right)$$

substitute $\frac{dy}{dx} = \frac{x^2}{y} \Rightarrow \frac{d^2y}{dx^2} = \frac{2x}{y} - \frac{x^2}{y^2}\left(\frac{x^2}{y}\right)$
$$= \frac{2x}{y} - \frac{x^4}{y^3}$$

Ex.3: If $y^2 = (\sin 2x)^4 + (\cos 2x)^4$, find y'.

sol.
$$2yy' = [4(\sin 2x)^3(\cos 2x)(2)] + [4(\cos 2x)^3(-\sin 2x)(2)]$$
$$2yy' = 8\sin^3 2x \cos 2x - 8\cos^3 2x \sin 2x$$
$$2yy' = 8\sin 2x \cos 2x(\sin^2 2x - \cos^2 2x)$$
$$[*-1] \Rightarrow -2yy' = 8\sin 2x \cos 2x(\cos^2 2x - \sin^2 2x)$$
$$-2yy' = 4\sin 4x \cos 4x$$
$$-2yy' = 2\sin 8x$$
$$\Longrightarrow y' = \frac{2\sin 8x}{-2y} = -\frac{\sin 8x}{y}$$

Ex.4: Find the tangent and the normal to the curve $x^2 - xy + y^2 = 7$ at the point $(-1, 2)$.



sol.
$$x^2 - xy + y^2 = 7$$
$$2x - (xy' + y(1)) + 2yy' = 0$$
$$2x - xy' - y + 2yy' = 0$$
$$y'(2y - x) = y - 2x$$
$$\Longrightarrow y' = \frac{y - 2x}{2y - x}$$ the slope of the tangent at any point on the curve.

$$\text{at } (-1,2) \Longrightarrow y' = \frac{2 - 2(-1)}{2(2) - (-1)} = \frac{4}{5} = \text{slope of the tangent at } (-1,2)$$

$$\text{slope of the normal line at } (-1,2) = \frac{-1}{4/5} = \frac{-5}{4}$$

$\Rightarrow$ The equation for the tangent is: $y - y_1 = m(x - x_1)$
$$y - 2 = \frac{4}{5}(x - (-1))$$
$$\Longrightarrow y = \frac{4}{5}x + \frac{14}{15}$$

The equation for the normal is: $y - 2 = \frac{-5}{4}(x - (-1))$
$$y - 2 = \frac{-5}{4}x - \frac{5}{4}$$
$$\Longrightarrow y = \frac{3}{4} - \frac{5}{4}x$$

: Exercises 3.6 / P.188 :

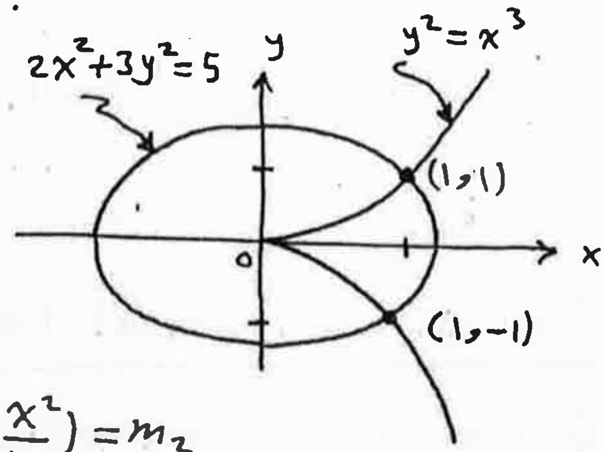
⑰ Find dy/dx for $y=\sqrt{1-\sqrt{x}}$

sol. $\frac{dy}{dx}=\frac{1}{2\sqrt{1-\sqrt{x}}} * \frac{-1}{2\sqrt{x}} = \frac{-1}{4\sqrt{x}\sqrt{1-\sqrt{x}}}$

⑳ Find dy/dx for $y=\sqrt{\sec 2x}$

sol. $\frac{dy}{dx}=\frac{1}{\cancel{2}\sqrt{\sec 2x}} * \sec 2x \tan 2x\,(\cancel{2}) = \frac{\sec 2x \tan 2x}{\sqrt{\sec 2x}}$

(44) Two curves are orthogonal at a point of intersection if their tangents there cross at right angle.
Show that the curves $2x^2+3y^2=5$ and $y^2=x^3$ are orthogonal at (1,1) and (1,-1).



sol. $2x^2+3y^2=5$

$4x+6yy'=0$

$\Rightarrow y'=\frac{-4x}{6y}=\frac{-2}{3}\left(\frac{x}{y}\right)=m_1$

Now $y^2=x^3$

$2yy'=3x^2 \Rightarrow y'=\frac{3x^2}{2y}=\frac{3}{2}\left(\frac{x^2}{y}\right)=m_2$

at the point (1,1): $m_1=\frac{-2}{3}\left(\frac{1}{1}\right)=\frac{-2}{3}$

$m_2=\frac{3}{2}\left(\frac{1^2}{1}\right)=\frac{3}{2}$

$m_1 * m_2 = \frac{-2}{3}*\frac{3}{2}=-1 \Rightarrow$ the two tangents are perpendicular

at the point (1,-1): $m_1=\frac{-2}{3}\left(\frac{1}{-1}\right)=\frac{2}{3}$

$m_2=\frac{3}{2}\left(\frac{1^2}{-1}\right)=-\frac{3}{2}$

$m_1 * m_2 = \frac{2}{3}*\frac{-3}{2}=-1 \Rightarrow$ the two tangents are perpendicular

$\Rightarrow$ The two curves are orthogonal at (1,1) and (1,-1).

Parametric Equations:

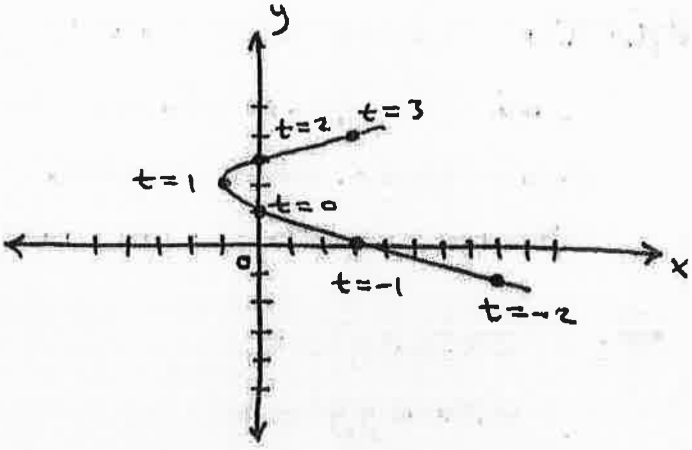
In parametric equations, the variables x and y are both given as functions of a third variable t (called parameter)

The parametric equation form is : $x = f(t)$, $y = g(t)$.

Ex. Sketch $x = t^2 - 2t$, $y = t + 1$.

sol.: Make a table with arbitrary values of t and find both x and y, then sketch from pairs of x and y.

t	x	y
-2	$(-2)^2 - 2(-2) = 8$	$-2+1=-1$
-1	$(-1)^2 - 2(-1) = 3$	$-1+1=0$
0	$(0)^2 - 2(0) = 0$	$0+1=1$
1	$(1)^2 - 2(1) = -1$	$1+1=2$
2	$(2)^2 - 2(2) = 0$	$2+1=3$
3	$(3)^2 - 2(3) = 3$	$3+1=4$



The First and Second Derivatives for the Parametric Equations:

① The First Derivative :

We have $x = f(t)$, $y = g(t)$, these two equations are equivalent to $y = F(x)$. And we want $F'(x)$ $\left[\text{or } \frac{dy}{dx}\right]$:

$y = F(x)$

$g(t) = F(f(t))$

differentiate with respect to t $\Longrightarrow g'(t) = F'(f(t)) \cdot f'(t)$

$\Longrightarrow g'(t) = F'(x) \cdot f'(t) \Longrightarrow F'(x) = \dfrac{g'(t)}{f'(t)}$

$$\Longrightarrow \boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}}$$ the 1st derivative for parametric equation.

② The Second Derivative:

The second derivative is $\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$, $y' = \frac{dy}{dx}$

or $$\boxed{\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}}$$ the 2nd derivative for parametric equation.

Example: If $x = \frac{t}{1-t}$, $y = t^2$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Solution: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$$\frac{dy}{dt} = 2t$$

$$\frac{dx}{dt} = \frac{(1-t)(1) - t(-1)}{(1-t)^2} = \frac{1-t+t}{(1-t)^2} = \frac{1}{(1-t)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2t}{\frac{1}{(1-t)^2}} = 2t(1-t)^2 = 2t(1-2t+t^2)$$

$$\Rightarrow \frac{dy}{dx} = 2t - 4t^2 + 2t^3$$

Now $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}(2t - 4t^2 + 2t^3) = 2 - 8t + 6t^2$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{2-8t+6t^2}{\frac{1}{(1-t)^2}} = (1-t)^2(2-8t+6t^2)$$

$$= (1-2t+t^2)(2-8t+6t^2) = 6t^4 - 20t^3 + 24t^2 - 12t + 2$$

The final step is finding $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of x by using $x = \frac{t}{1-t} \Rightarrow t = \frac{x}{x+1}$. [Substitute t in terms of x].

L' Hopital's Rule : The L' Hopital's Rule says:

If $\lim\limits_{x \to c} \frac{f(x)}{g(x)}$ has indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and that

$\lim\limits_{x \to c} \frac{f'(x)}{g'(x)} = L$ (or $\mp\infty$). Then $\lim\limits_{x \to c} \frac{f(x)}{g(x)} = \lim\limits_{x \to c} \frac{f'(x)}{g'(x)}$.

c: any number or $\mp\infty$.

Ex.1 : Find $\lim\limits_{x \to 0} \frac{1 - \cos x}{\sin x} \longrightarrow \frac{1-1}{0} = \frac{0}{0}$

Use L' Hopital's rule $\Rightarrow \lim\limits_{x \to 0} \frac{1 - \cos x}{\sin x} = \lim\limits_{x \to 0} \frac{\frac{d}{dx}(1 - \cos x)}{\frac{d}{dx}(\sin x)}$

$$= \lim_{x \to 0} \frac{\sin x}{\cos x} = \lim_{x \to 0} \tan x = \tan 0 = 0$$

Ex.2 : $\lim\limits_{x \to 0} \frac{x - \sin x}{x^2} \longrightarrow \frac{0}{0}$

L' Hopital $\Longrightarrow \lim\limits_{x \to 0} \frac{x - \sin x}{x^2} = \lim\limits_{x \to 0} \frac{1 - \cos x}{2x} \longrightarrow \frac{0}{0}$

L' Hopital again $\Longrightarrow \lim\limits_{x \to 0} \frac{\sin x}{2} = \frac{0}{2} = 0$

[Note : L' Hopital's Rule can be used repeatedly].

Ex.3 : $\lim\limits_{x \to \infty} \frac{2x^3 + 3x^2 + 1}{x^2 + 4} \longrightarrow \frac{\infty}{\infty}$

L' Hopital $\Longrightarrow \lim\limits_{x \to \infty} \frac{6x^2 + 6x}{2x} = \lim\limits_{x \to \infty} \frac{2x(3x+3)}{2x} = \lim\limits_{x \to \infty} 3x + 3 = \infty$

Ex.4 : $\lim\limits_{x \to \frac{\pi}{2}} (\sec x - \tan x) \longrightarrow \infty - \infty$

circumvent the problem $\Rightarrow \lim\limits_{x \to \frac{\pi}{2}} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)$

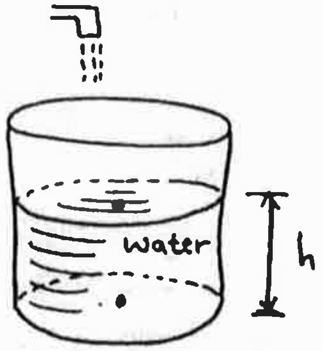
$$= \lim_{x \to \frac{\pi}{2}} \frac{1 - \sin x}{\cos x} \longrightarrow \frac{0}{0}$$

L' Hopital $\Longrightarrow \lim\limits_{x \to \frac{\pi}{2}} \frac{-\cos x}{-\sin x} = \frac{0}{1} = 0$

CHAPTER FOUR

APPLICATIONS OF DERIVATIVES

4.1 : Related Rates of Changes:



If water is pumped into a tank, both the volume of the water (V) and the height of water level (h) are increasing and their rates of increase are related to each other.

The rate of change in quantity is the derivative of this quantity with respect to the time t.

In this example:

$\frac{dV}{dt}$ is the rate of change in the volume of water.

$\frac{dh}{dt}$ is the rate of change in the water level.

$\frac{dV}{dt}$ and $\frac{dh}{dt}$ are called related rates.

In related rates problems, we compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured).

Steps for solution:

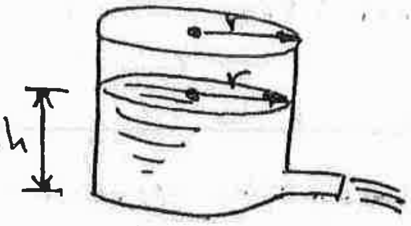
1- Write an equation that relates the variable quantities.

2- Differentiate both sides with respect to the time t.

3- Substitute the given information and solve for the unknown rate.

Ex.1: How fast does the water level drop when a cylindrical tank is drained at the rate of $5\, m^3/sec$

Sol. let V = volume of water



$\frac{dh}{dt}$?

$$V = \pi r^2 h \qquad [r : \text{constant}]$$

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$$

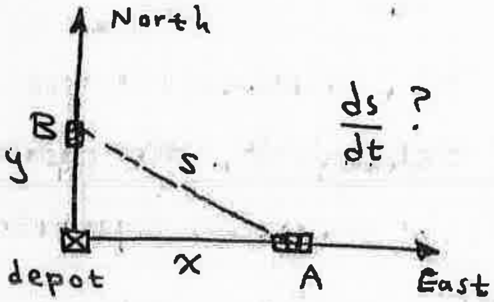
$$-5 = \pi r^2 \frac{dh}{dt}$$

$$\Rightarrow \boxed{\frac{dh}{dt} = -\frac{5}{\pi r^2}} \; m/sec$$

[the minus sign means that the water level is decreasing.]

If $r = 1\, m \Rightarrow \frac{dh}{dt} = -\frac{5}{\pi (1)^2} = \frac{-5}{\pi}$ m/sec.

Ex.2: Two cars leave a depot. Car A traveling east at 40 mph and car B traveling north at 30 mph. How fast is the distance between the cars changing after 6 minutes?



$\frac{ds}{dt}$?

Sol.

$$s^2 = x^2 + y^2$$

$$2s \frac{ds}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

$$\Rightarrow \frac{ds}{dt} = \frac{1}{s} \left(x \frac{dx}{dt} + y \frac{dy}{dt} \right)$$

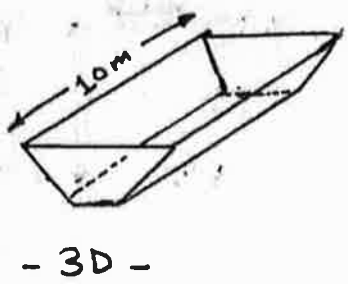
After 6 min. : $x = 40 \frac{mile}{hr} * \frac{6}{60} hr = 4$ mile $\quad \left[velocity = \frac{distance}{time} \right]$

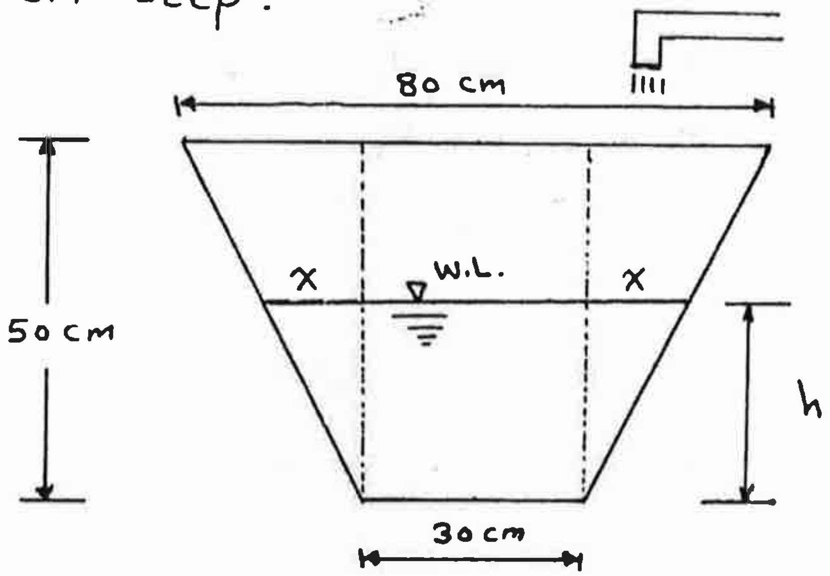
$y = 30 * \frac{6}{60} = 3$ mile

$s = \sqrt{x^2 + y^2} = \sqrt{16 + 9} = 5$ mile

$$\Rightarrow \frac{ds}{dt} = \frac{1}{5}\left[(4*40) + (3*30)\right] = \frac{1}{5}(250) = 50 \text{ mph}.$$

Ex.3: A water trough is 10 m long and a cross section has the shape of isosceles trapezoid as shown in the figure below. If the trough is being filled with water at the rate of 0.2 m^3/min, how fast is the water level rising when the water is 30 cm deep?





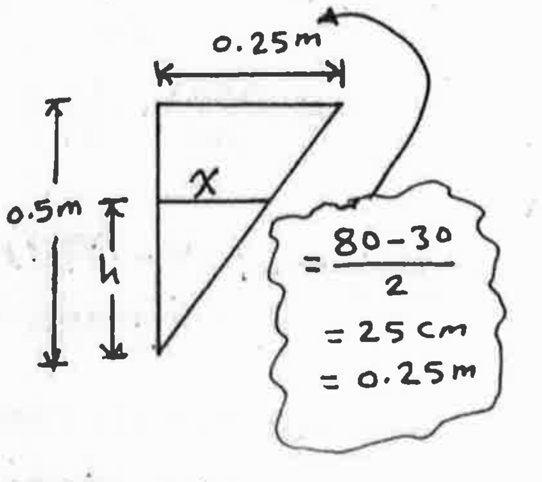
Sol. Let V = volume of water.

$$V = \underbrace{\frac{(0.3+2x)+0.3}{2} * h}_{\text{cross section area of water}} * \underbrace{10}_{\text{trough length}} = (0.6+2x) * 5h$$

$$\Rightarrow V = 3h + 10xh$$

From similarity of triangles:

$$\frac{x}{h} = \frac{0.25}{0.50} = \frac{1}{2} \Rightarrow x = \frac{h}{2}$$



$$\text{substitute} \Rightarrow V = 3h + 10\left(\frac{h}{2}\right) * h$$

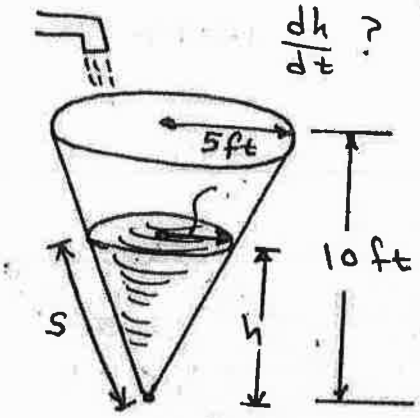
$$\Rightarrow V = 3h + 5h^2$$

$$\frac{dV}{dt} = 3\frac{dh}{dt} + 10h\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{dV/dt}{3+10h}$$

$$h = 30 \text{ cm} = 0.3m, \quad \frac{dV}{dt} = 0.2 \Rightarrow \frac{dh}{dt} = \frac{0.2}{3+10(0.3)} = \frac{0.2}{6} = \frac{1}{30} \text{ m/min}$$

Exercises 4.1 / P. 222 :

(22) Water runs into conincal tank, with radius of base of 5 ft and height 10 ft, at the rate of 9 ft^3/min. How fast is the water level rising when the water is 6 ft deep?

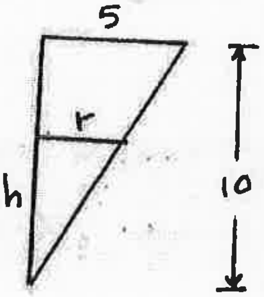


Sol. V = volume of water

$$V = \frac{1}{3}\pi r^2 h$$

From similarity of triangles:

$$\frac{r}{h} = \frac{5}{10} = \frac{1}{2} \Rightarrow r = \frac{h}{2}$$



$$\Rightarrow V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 * h = \frac{\pi}{3} * \left(\frac{h^2}{4}\right) * h$$

$$\Rightarrow V = \frac{\pi}{12} h^3$$

$$\frac{dV}{dt} = 3 * \frac{\pi}{12} h^2 . \frac{dh}{dt} = \frac{\pi}{4} h^2 . \frac{dh}{dt}$$

at $h = 6$, $\frac{dV}{dt} = 9 \Rightarrow 9 = \frac{\pi}{4}(6)^2 . \frac{dh}{dt}$

$$\Rightarrow \frac{dh}{dt} = \frac{1}{\pi} \text{ ft/min.} \quad \approxeq 0.318 \text{ ft/min.}$$

Addition : How fast is the surface area of the water changing when the water is 6 ft deep?

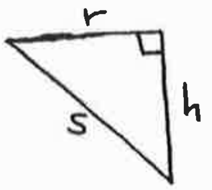
Let A = total surface area of water

= area of base + side area of the cone

$= \pi r^2 + \pi r s$

>>>

From Pythagorean theorem:



$$s = \sqrt{r^2+h^2} = \sqrt{(\tfrac{h}{2})^2+h^2} = \sqrt{\tfrac{h^2}{4}+h^2}$$

$$= \sqrt{\frac{5h^2}{4}} = \frac{\sqrt{5}}{2} h$$

$$\Rightarrow A = \pi(\tfrac{h}{2})^2 + \pi(\tfrac{h}{2})(\tfrac{\sqrt{5}}{2}h).$$

$$\Rightarrow A = \frac{\pi}{4}h^2 + \frac{\sqrt{5}}{4}\pi h^2 .$$

$$\frac{dA}{dt} = \frac{\pi}{2} h \frac{dh}{dt} + \frac{\sqrt{5}}{2}\pi h \frac{dh}{dt}$$

at $h = 6$., $\frac{dh}{dt} = \frac{1}{\pi}$

$$\Rightarrow \frac{dA}{dt} = \left(\frac{\pi}{2} * 6 * \frac{1}{\pi}\right) + \left(\frac{\sqrt{5}}{2} * \pi * 6 * \frac{1}{\pi}\right) = 3 + 3\sqrt{5} \text{ ft}^2/\text{min}.$$

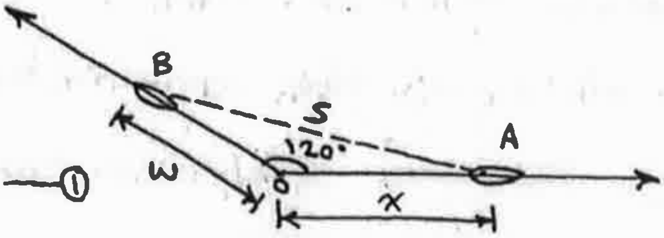
$$\simeq 9.7 \text{ ft}^2/\text{min}.$$

(31) Two ships are steaming straight away from a point O along routes that make a 120° angle. Ship A moves at 14 knots. Ship B moves at 21 knots. How fast are the ships moving apart when OA = 5 and OB = 3 nautical miles? = 6080 ft

mile = 5280 ft

$\text{Knots} = \frac{\text{no. mile}}{\text{hr}}$

sol. Use cosine law:



$$s^2 = x^2 + w^2 - 2xw \cos 120$$

$$\cos 120^\circ = -\tfrac{1}{2} \Rightarrow s^2 = x^2 + w^2 + xw \text{ —— ①}$$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2w\frac{dw}{dt} + x\frac{dw}{dt} + w\frac{dx}{dt} \text{ —— ②}$$

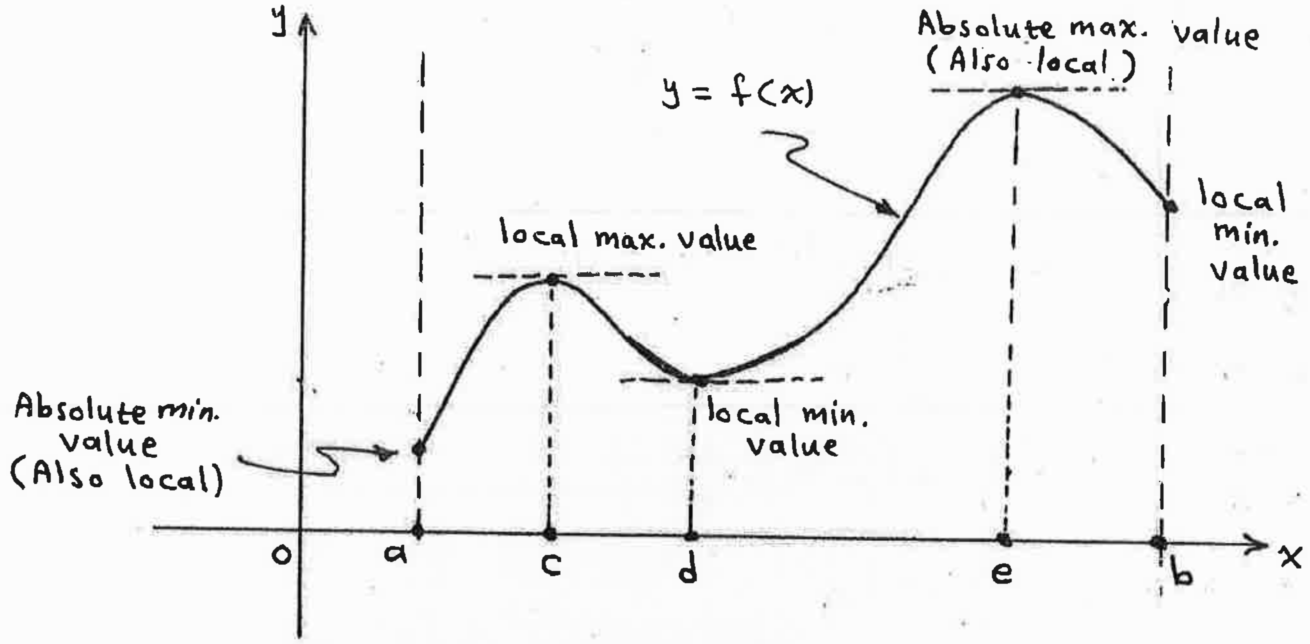
when $x = 5$, $w = 3$ $\Rightarrow$ $s^2 = (5)^2 + (3)^2 + 5(3) = 49$ [from eq.①]

$\Rightarrow s = 7$ nautical mile

$$\xRightarrow{eq②} 2*7*\frac{ds}{dt} = (2*5*14) + (2*3*21) + (5*21) + (3*14)$$

$$\Longrightarrow \frac{ds}{dt} = 29.5 \text{ knots}.$$

4.2 : Maxima, Minima, and Mean Value Theorem :



Definitions:

① Local maximum value : is the maximum value of f(x) for all x in some open interval about c.

② Local minimum value : is the minimum value of f(x) for all x in some open interval about d.

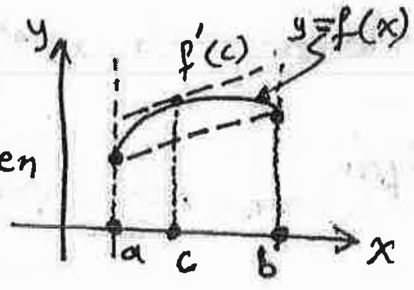
③ Absolute maximum value : is the maximum value of f(x) for all x in the domain [a, b]. [at x = e in the fig.]

④ Absolute minimum value : is the minimum value of f(x) for all x in the domain [a, b]. [at x = a in the fig.]

Note : For the endpoints (a & b), the local maximum and local minimum considered for half open interval.

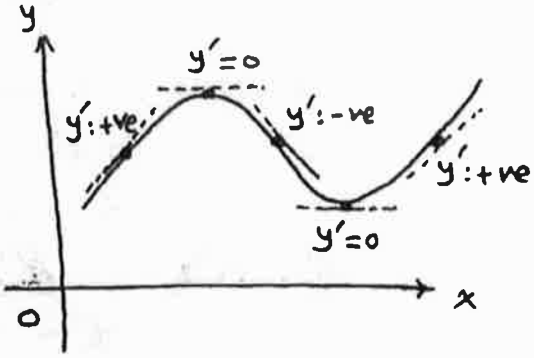
The mean value theorem :

If $y = f(x)$ is continuous at every point of interval [a, b] and differentiable, then there is at least one number c between a & b in which $f'(c) = \frac{f(b) - f(a)}{b - a}$.



4.3 : Curve Sketching with y' and y'' :

First derivative :



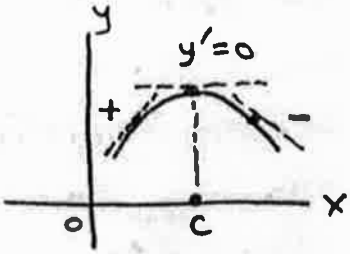
The first derivative of a function gives the slope of the tangent to the curve of the function.

We have three cases for y' :

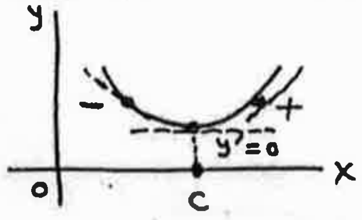
1- If y' is positive ⇒ the function is increasing.

2- If y' is negative ⇒ the function is decreasing.

3- If $y' = 0$ at $x = c$ ⇒ the function has local maximum or local minimum value at $x = c$ if the signs of y' are different before and after the point c.



(maximum)



(minimum)

Notes :

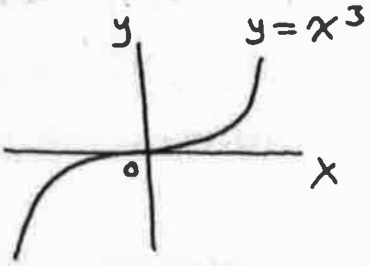
1- A curve may have a horizontal tangent without having local max. or min. value.

For example : $y = x^3$

$y' = 3x^2$

at $x = 0 \Rightarrow y' = 0$

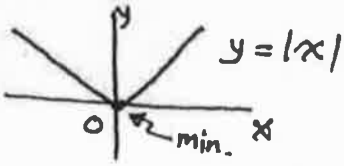
(no local max. or min. at $x=0$)



2- A curve may have local max. or min. value without having horizontal tangent.

For example : $y = |x|$ is not differentiable at $x = 0$

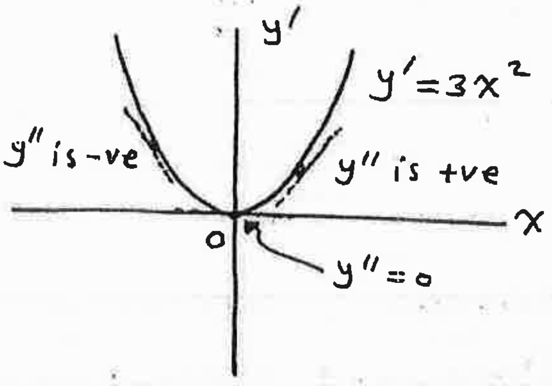
→ no tangent.

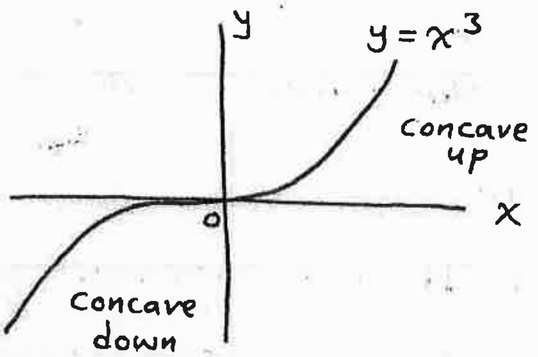


Second derivative :

The second derivative gives the concavity of the curve.

For example, if we have the function $y=x^3$:



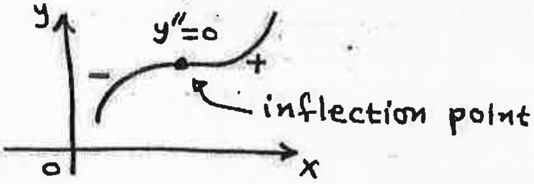


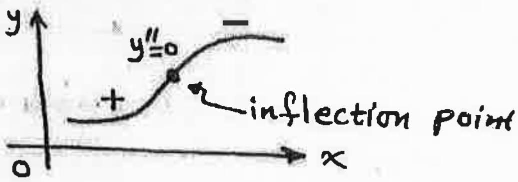
We have three cases for y'':

1- If y'' is positive $\Rightarrow$ the curve is concave up $\smile$.

2- If y'' is negative $\Rightarrow$ the curve is concave down $\frown$.

3- If $y''=0$ at $x=c \Rightarrow$ the point at $x=c$ is inflection point if the signs of y'' are different before and after c.



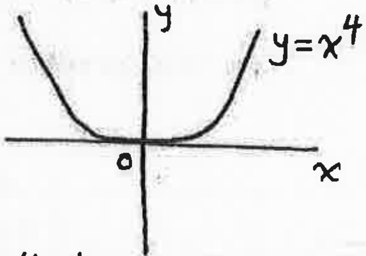


Notes :

1- y'' may be equals zero at a point, but the point is not inflection point.

Ex. $y=x^4 \Rightarrow y'=4x^3 \Rightarrow y''=12x^2$

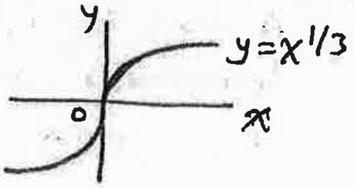
at $x=0 \Rightarrow y''=0$ (but at $x=0$ no inflection point)



2- The curve may have inflection point but y'' does not exist.

Ex. $y=x^{1/3} \Rightarrow y'=\frac{1}{3}x^{-2/3} \Rightarrow y''=\frac{-2}{9}x^{-5/3}$

$\Rightarrow y''=\frac{-2}{9}\left(\frac{1}{x^{5/3}}\right)$ [not exist at $x=0$]



Notes : ① Critical points is the point where $y'=0$ or y' not defined.

② If $y'=0$ and y'' is positive at a point ⇒ min. point.

③ If $y'=0$ and y'' is negative at a point ⇒ max. point.

Examples :

Ex.1 : Graph $y=x^3-3x^2+4$

Sol. ① $y'=3x^2-6x$

$y'=0 \Rightarrow 3x^2-6x=0 \Rightarrow 3x(x-2)=0$

$\Rightarrow 3x=0 \Rightarrow \boxed{x=0}$ and $x-2=0 \Rightarrow \boxed{x=2}$

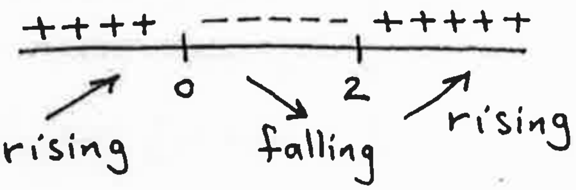
at $x=0 \Rightarrow y=(0)^3-3(0)^2+4=4$

⇒ $(0,4)$ is max. point.

at $x=2 \Rightarrow y=2^3-3(2)^2+4=0$

⇒ $(2,0)$ is min. point.

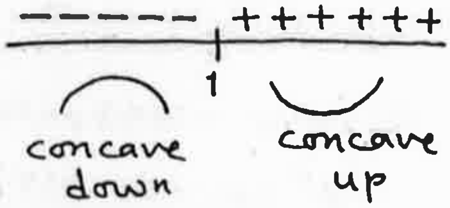
sign of y' :



② $y''=6x-6$

$y''=0 \Rightarrow 6x-6=0$

$\Rightarrow 6x=6 \Rightarrow \boxed{x=1}$



at $x=1 \Rightarrow y=1^3-3(1)^2+4=2$

⇒ $(1,2)$ is inflection point.

③ Intercepts :

For x-intercept : $y=0$

$\Rightarrow x^3-3x^2+4=0$

$(x+1)(x^2-4x+4)=0$

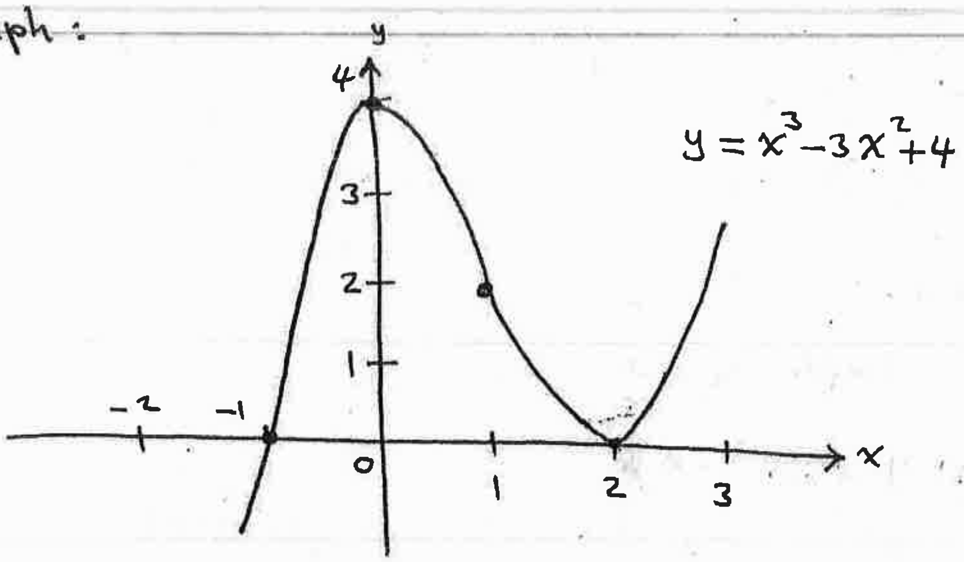
$x+1=0 \Rightarrow \boxed{x=-1}$

$x^2-4x+4=0 \Rightarrow (x-2)(x-2)=0 \Rightarrow \boxed{x=2}$

For y-intercept : $x=0 \Rightarrow \boxed{y=4}$

$$\begin{array}{r} x^2-4x+4 \\ x+1 \,\overline{)\, x^3-3x^2+4} \\ \mp x^3 \mp x^2 \downarrow \\ \hline -4x^2+4 \\ \pm 4x^2 \pm 4x \\ \hline 4x+4 \\ \mp 4x \mp 4 \\ \hline 0 \quad 0 \end{array}$$

④ The graph:



Ex.2 : Sketch $y = \sin x + \cos x$ from $x = \frac{-\pi}{4}$ to $\frac{3\pi}{4}$.

Sol. ① $y' = \cos x - \sin x$

$y' = 0 \Rightarrow \sin x = \cos x$

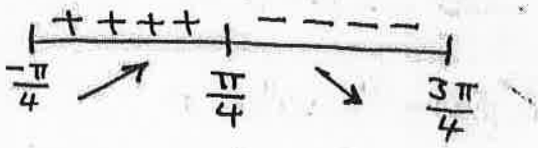
$[\div \cos x] \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}$

at $x = \frac{\pi}{4} \Rightarrow y = \sin\frac{\pi}{4} + \cos\frac{\pi}{4}$

$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$

$\Rightarrow (\frac{\pi}{4}, \sqrt{2})$ is max. point.

sign of y':



② $y'' = -\sin x - \cos x$

$y'' = 0 \Rightarrow \sin x = -\cos x$

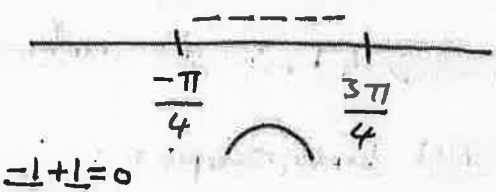
$[\div \cos x] \Rightarrow \tan x = -1$

$\Rightarrow x = \frac{-\pi}{4}$ & $x = \frac{3\pi}{4}$

at $x = \frac{-\pi}{4} \Rightarrow y = \sin\frac{-\pi}{4} + \cos\frac{-\pi}{4} = \frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 0$

at $x = \frac{3\pi}{4} \Rightarrow y = 0 \quad [\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0]$

sign of y'':



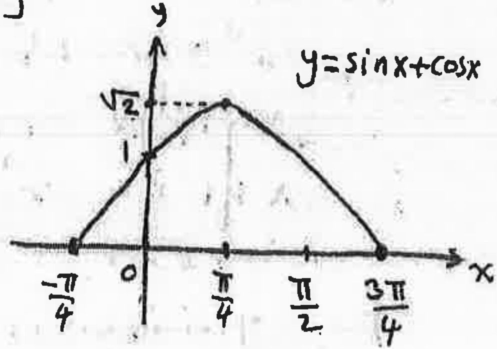
③ Intercepts:

y-intercept : $x = 0 \Rightarrow y = 1$

x-intercept : $y = 0 \Rightarrow \sin x + \cos x = 0$

$\Rightarrow \sin x = -\cos x \Rightarrow \tan x = -1$

$\Rightarrow x = \frac{-\pi}{4}$ & $x = \frac{3\pi}{4}$



(The graph)

Ex.3: Find x-intercept and y-intercept for the function $y = x^3-2x^2+1$.

$$\begin{array}{r|l} & x^2-x-1 \\ \hline x-1 & x^3-2x^2+1 \\ & \mp x^3 \pm x^2 \\ \hline & -x^2+1 \\ & \pm x^2 \mp x \\ \hline & -x+1 \\ & \pm x \mp 1 \\ \hline & 0\quad 0 \end{array}$$

Sol. x-intercept: $y=0$

$$x^3-2x^2+1=0$$

$$(x-1)(x^2-x-1)=0$$

$$x-1=0 \Rightarrow \boxed{x=1}$$

and $x^2-x-1=0 \Longleftrightarrow ax^2+bx+c=0$

$$x=\frac{-b\mp\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{1\mp\sqrt{1+4}}{2}$$

$$\Rightarrow \boxed{x=\frac{1+\sqrt{5}}{2}} \text{ and } \boxed{x=\frac{1-\sqrt{5}}{2}}$$

y-intercept: $x=0$

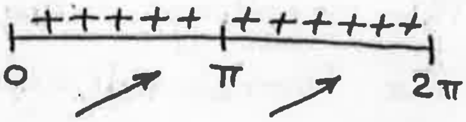
$$\Rightarrow y=(0)^3-2(0)^2+1=1 \Rightarrow \boxed{y=1}$$

Ex.4: Graph $y=x+\sin x$, $0\leqslant x\leqslant 2\pi$.

Sol. $y'=1+\cos x$

$y'=0 \Rightarrow \cos x=-1 \Rightarrow x=\pi$

sign of y':



$y''=-\sin x$

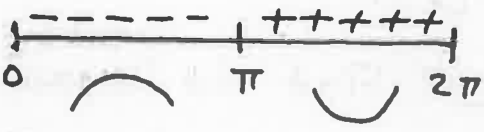
$y''=0 \Rightarrow \sin x=0$

$\Rightarrow x=0,\ \pi,$ and 2π

at $x=\pi \Rightarrow y=\pi+\sin\pi=\pi$

$\Rightarrow (\pi,\pi)$ is inflection point.

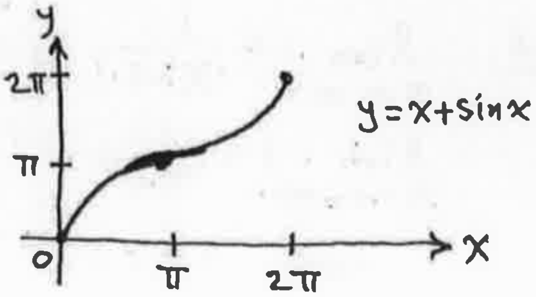
sign of y'':



ounds: at $x=0 \Rightarrow y=0+\sin 0=0$

at $x=2\pi \Rightarrow y=2\pi+\sin 2\pi=2\pi$

$\Rightarrow$ the points: $(0,0)$ & $(2\pi,2\pi)$



4.4: Graphing Rational Functions:

Asymptotes: If the distance between the graph and some fixed line approaches zero as the graph moves farther and farther from the origin, we say that this line is asymptote of the graph.

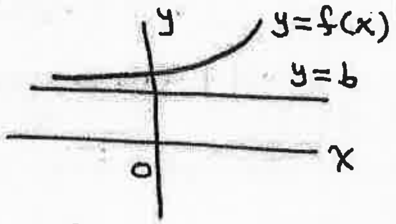
There are four types of asymptotes:

1- Horizontal asymptotes.
2- Vertical asymptotes.
3- Oblique (or slant) asymptotes.
4- Curved asymptotes.

① Horizontal asymptotes:

The line $y=b$ is horizontal asymptote of the graph $y=f(x)$ if either:

$$\lim_{x\to\infty} f(x)=b \quad \text{or} \quad \lim_{x\to-\infty} f(x)=b$$



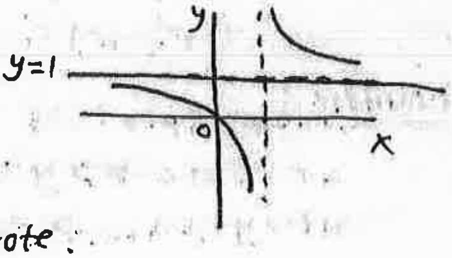
To find the horizontal asymptotes for a function, take the limits of the function as x approaches ∞ and $-\infty$. The resulting values represent b_1 and b_2.
The horizontal asymptotes will be $y=b_1$ and $y=b_2$.
If $b_1=b_2$ (this is a common case), then the horizontal asymptote becomes $y=b$. $[b=b_1=b_2]$

Ex.1: Find the horizontal asymptotes for $y=1+\dfrac{1}{x-1}$.

sol.
$$\lim_{x\to\infty} 1+\frac{1}{x-1} = 1+\frac{1}{\infty}=1$$

$$\lim_{x\to-\infty} 1+\frac{1}{x-1} = 1+\frac{1}{-\infty}=1$$

$\Longrightarrow$ $y=1$ is horizontal asymptote.



Ex.2: Find the horizontal asymptotes for the function:

$$y = \frac{\sqrt{2x^2+1}}{3x-5}$$

Sol. $\lim\limits_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim\limits_{x\to\infty} \frac{\sqrt{\frac{2x^2}{x^2}+\frac{1}{x^2}}}{\frac{3x}{x}-\frac{5}{x}} = \frac{\sqrt{2+0}}{3-0} = \frac{\sqrt{2}}{3}$

$$\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to-\infty} \frac{\frac{1}{x}\sqrt{2x^2+1}}{\frac{1}{x}(3x-5)} = \lim_{x\to-\infty} \frac{\frac{1}{-\sqrt{x^2}}\sqrt{2x^2+1}}{\frac{1}{x}(3x-5)}$$

* Why $x = -\sqrt{x^2}$?

Since $\sqrt{x^2} = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$. And we have $x \to -\infty$

Then $\sqrt{x^2} = -x \Rightarrow x = -\sqrt{x^2} \Rightarrow \frac{1}{x} = \frac{1}{-\sqrt{x^2}}$

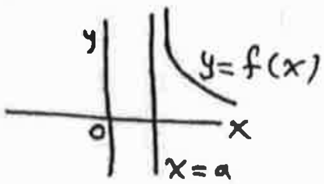
$$\Rightarrow \lim_{x\to-\infty} \frac{-\sqrt{\frac{2x^2}{x^2}+\frac{1}{x^2}}}{\frac{3x}{x}-\frac{5}{x}} = \frac{-\sqrt{2}}{3}$$

$\Rightarrow$ The horizontal asymptotes are $y = \frac{\sqrt{2}}{3}$ and $y = \frac{-\sqrt{2}}{3}$.

② Vertical Asymptotes:

The line $x = a$ is vertical asymptote if:

either $\lim\limits_{x\to a^+} f(x) = \mp\infty$ or $\lim\limits_{x\to a^-} f(x) = \mp\infty$.



To find the vertical asymptotes for a function, find the values of x that make the denominator equals zero and check that the limit of a function goes to (∞ or $-\infty$) as x approaches (a^+ or a^-).

Ex.1: Find the vertical asymptotes for $y = \frac{-8}{x^2-4}$.

sol. $x^2-4=0 \Longrightarrow x^2=4 \Longrightarrow x=2$ and $x=-2$.

Check: $\lim\limits_{x\to 2^+} \frac{-8}{x^2-4} = \frac{-8}{0^+} = -\infty$

$\lim\limits_{x\to -2^+} \frac{-8}{x^2-4} = \frac{-8}{0^-} = \infty$

$\Longrightarrow x=2$ and $x=-2$ are vertical asymptotes

Ex.2: Find the vertical asymptotes for $y = \frac{x^2+x-6}{x^2-4}$.

sol. $x^2-4=0 \Rightarrow x^2=4 \Rightarrow x=2$ and $x=-2$.

Check: $\lim\limits_{x\to 2^+} \frac{x^2+x-6}{x^2-4} = \frac{0}{0} \neq \infty$ or $-\infty$.

$\lim\limits_{x\to 2^-} \frac{x^2+x-6}{x^2-4} = \frac{0}{0} \neq \infty$ or $-\infty$.

$\Longrightarrow x=2$ is not vertical asymptote.

Now $\lim\limits_{x\to -2^+} \frac{x^2+x-6}{x^2-4} = \frac{-4}{0^-} = \infty$

$\Longrightarrow x=-2$ is the vertical asymptote.

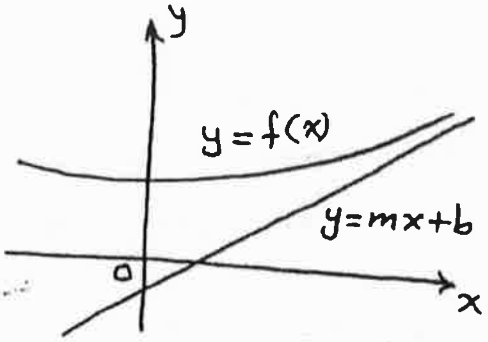
Note: If we circumvent the problem:

$$y = \frac{x^2+x-6}{x^2-4} = \frac{\cancel{(x-2)}(x+3)}{\cancel{(x-2)}(x+2)} = \frac{x+3}{x+2}.$$

$x+2=0 \Longrightarrow x=-2$ is vertical asymptote.

③ Oblique (or Slant) asymptotes:

When $\lim\limits_{x \to \mp\infty} [f(x) - (mx+b)] = 0$ then the line $y = mx+b$ is oblique asymptote for the function $f(x)$.



For the rational functions:
If the degree of numerator is one greater than the degree of denominator, the graph has an oblique asym.
To find the oblique asymptote, divide the numerator over the denominator (by long division).
The result represents the oblique asymptote.

Ex. Find the oblique asymptote for $y = \dfrac{x^2-3}{2x-4}$.

sol. deg. of numerator − deg. of denominator $= 2-1 = 1$

Use long division:

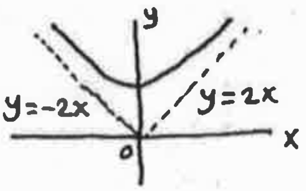
$$\begin{array}{r|l} & \frac{x}{2}+1 \\ \hline 2x-4 & x^2-3 \\ & \mp x^2 \pm 2x \\ \hline & \quad 2x-3 \\ & \quad \mp 2x \pm 4 \\ \hline & \qquad 1 \end{array}$$

$$\Rightarrow \quad y = \underbrace{\frac{x}{2}+1}_{\text{result}} + \underbrace{\frac{1}{2x-4}}_{\text{remainder}}$$

$\Rightarrow \quad y = \frac{x}{2}+1$ is the oblique asym.

Note:

A function may have oblique asymptotes but it is not rational function.
For example: the function $y = \sqrt{4x^2+9}$ has two oblique asymptotes $y = 2x$ & $y = -2x$.



④ Curved asymptotes :

If the degree of numerator is more than one greater than the degree of denominator, the asymptote becomes curved.

Use the long division to find the curved asymptote.

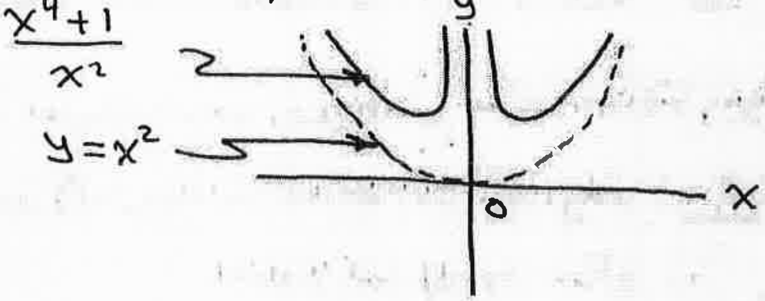
Ex. Find the curved asymptote for the function:

$$y = \frac{x^4+1}{x^2}.$$

Sol. $y = \underbrace{x^2}_{} + \frac{1}{x^2}$

$$\begin{array}{r|l} & x^2 \\ \hline x^2 & x^4+1 \\ & \mp x^4 \downarrow \\ \hline & 0 \quad 1 \end{array}$$

$\Rightarrow y = x^2$ is the curved asymptote.



The result of division represents the curved asymptote. [$y=x^2$ in this example].

Dominant terms :

Like the asymptotes, the dominant terms is used to graph the rational functions.

Ex. $y = \frac{x+3}{x+2}$

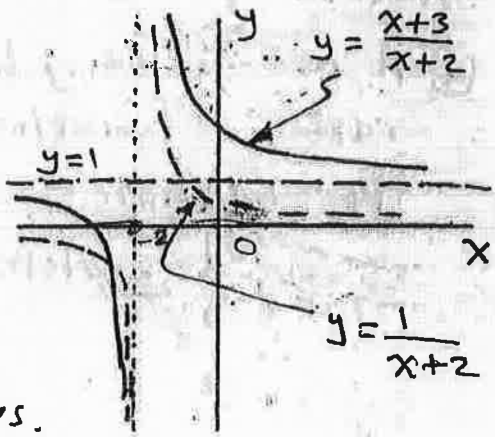
$$\begin{array}{r|l} & 1 \\ \hline x+2 & x+3 \\ & \mp x \mp 2 \\ \hline & 1 \end{array}$$

Long division $\Rightarrow y = 1 + \frac{1}{x+2}$

$y=1$ for $x \to \mp\infty$

$y = \frac{1}{x+2}$ for $x \to -2$

(1) and $\left(\frac{1}{x+2}\right)$ are dominant terms.



Note: Symmetry + derivatives + asymptotes + dominant terms give effective graph for rational functions.

Examples for graphing rational functions:

Ex.1: Use symmetry, first derivative, second derivative and asymptotes to graph the function:

$$y = \frac{x^2}{x^2-1}.$$

Sol. ① Symmetry:

$$f(-x) = \frac{(-x)^2}{(-x)^2-1} = \frac{x^2}{x^2-1} = f(x)$$

$\Rightarrow$ the function is even function (symmetric about y-axis).

② 1^{st} derivative:

$$\begin{array}{r} 1 \\ x^2-1 \overline{\smash{)}\; x^2 \quad} \\ \mp x^2 \pm 1 \\ \hline 1 \end{array}$$

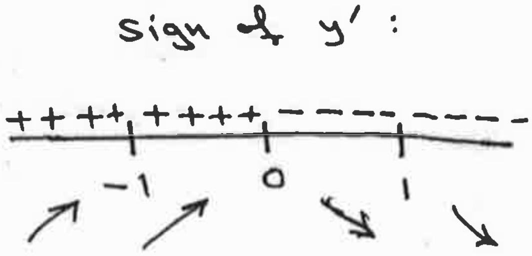
$$y = 1 + \frac{1}{x^2-1}$$

$$y' = 0 - \frac{2x}{(x^2-1)^2} = -\frac{2x}{(x^2-1)^2} = 2x\left(\frac{-1}{(x^2-1)^2}\right)$$

$$y' = 0 \Rightarrow \frac{-1}{(x^2-1)^2} \neq 0 \quad \text{and} \quad 2x = 0 \Rightarrow x = 0$$

The values of x that make the 1^{st} derivative (y') not defined are: $x^2-1=0 \Rightarrow x^2 = 1 \Rightarrow x = 1$ & $x = -1$.

Divide the x-axis from the values of x in which $y'=0$ and y' not defined.



$$\text{at } x = 0 \Rightarrow y = 1 + \frac{1}{(0)^2-1} = 1-1 = 0$$

$\Rightarrow$ the point $(0,0)$ is max. point.

③ 2^{nd} derivative:

$$y' = \frac{-2x}{(x^2-1)^2} \Rightarrow y'' = \frac{(x^2-1)^2(-2)-(-2x)[2(x^2-1)(2x)]}{(x^2-1)^4}$$

$$\Rightarrow y'' = \frac{-2(x^2-1)^2+8x^2(x^2-1)}{(x^2-1)^4} = \frac{\cancel{(x^2-1)}[-2(x^2-1)+8x^2]}{(x^2-1)^{\cancel{4}3}}$$

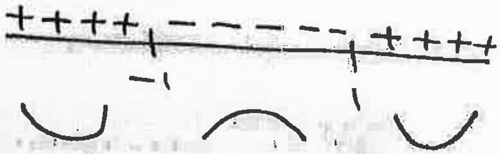
$$\Rightarrow y'' = \frac{6x^2+2}{(x^2-1)^3} = (6x^2+2)\left(\frac{1}{(x^2-1)^3}\right)$$

$y''=0 \Rightarrow \frac{1}{(x^2-1)^3} \neq 0$ and $6x^2+2=0 \Rightarrow x^2 \neq \frac{-1}{3}$

$\Rightarrow y'' \neq 0$

The values of x that make y'' not defined are $x^2-1=0$
$\Rightarrow x^2=1 \Rightarrow x=1$ & $x=-1$

sign of y''



$x=-1$ & $x=1$ out of domain ⟹ no inflection points.

④ Asymptotes:

Horizontal asym.: $\lim_{x\to\infty} 1+\frac{1}{x^2-1} = 1+\frac{1}{\infty} = 1+0=1$

$\lim_{x\to-\infty} 1+\frac{1}{x^2-1} = 1+\frac{1}{-\infty} = 1+0=1$

⟹ $y=1$ is horizontal asymptote.

Vertical asym.: $x^2-1=0 \Rightarrow x^2=1 \Rightarrow x=1$ & $x=-1$

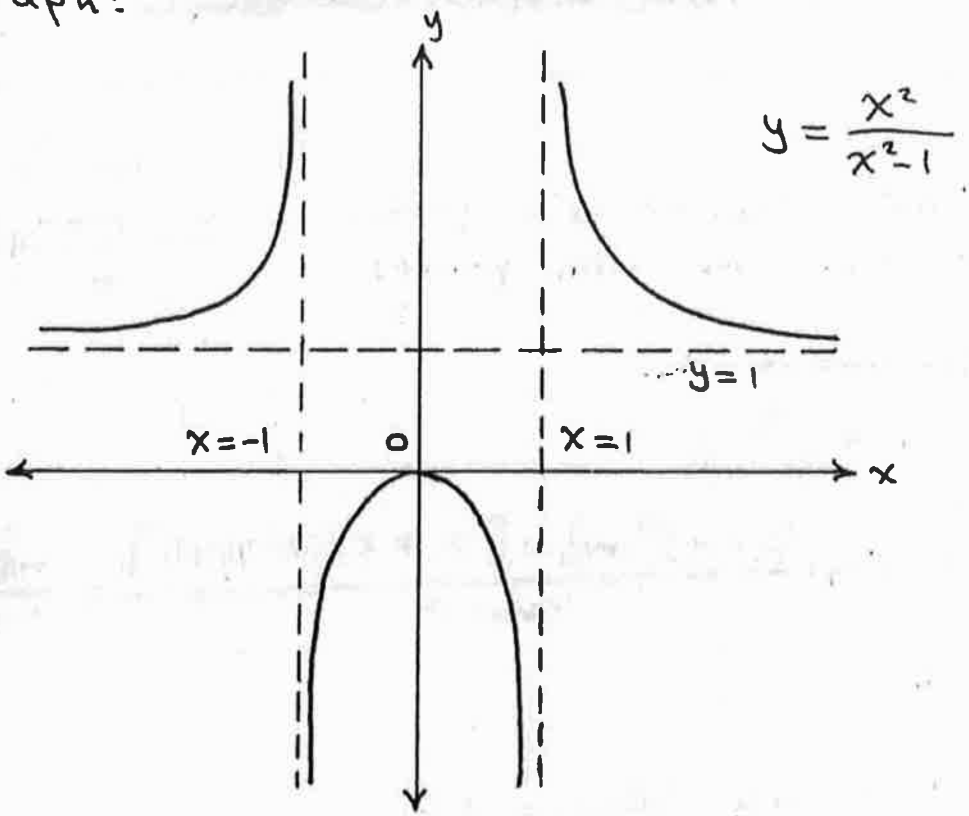
Check: $\lim_{x\to1^+} 1+\frac{1}{x^2-1} = 1+\infty=\infty$

$\lim_{x\to-1^+} 1+\frac{1}{x^2-1} = 1-\infty=-\infty$

⟹ $x=1$ & $x=-1$ are vertical asymptotes.

Note: The horizontal asymptote may be determined by using L'Hopital's rule to $y=\frac{x^2}{x^2-1}$.

⑤ The graph:



Ex.2 : Use first derivative, second derivative, and the asymptotes to graph $y = \frac{x^2-4}{x-1}$.

sol. ① Asymptotes:

$$\begin{array}{r} x+1 \\ x-1\,\overline{)\,x^2-4} \\ \mp x^2 \pm x \\ \hline x-4 \\ \mp x \pm 1 \\ \hline -3 \end{array}$$

$y = x+1 - \frac{3}{x-1}$

$y = x+1$ is oblique asymptote.

Vertical : $x-1=0 \Rightarrow x=1$

check : $\lim_{x \to 1^+} x+1-\frac{3}{x-1} = -\infty$

$\Rightarrow x=1$ is vertical asymptote.

② 1^{st} derivative:

$y' = 1+0+\frac{3}{(x-1)^2} = 1+\frac{3}{(x-1)^2}$

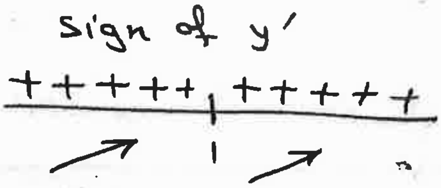
$y'=0 \Rightarrow \frac{3}{(x-1)^2} = -1 \Rightarrow (x-1)^2 = -3 \Rightarrow x^2-2x+4=0$

$x = \frac{-b \mp \sqrt{b^2-4ac}}{2a} = \frac{-(-2) \mp \sqrt{(-2)^2-4(1)(4)}}{2(1)} = \frac{2 \mp \sqrt{-12}}{2}$ (not defined)

$\Rightarrow y' \neq 0$

the value of x that makes y' not defined is $x-1=0 \Rightarrow x=1$

no max. or min. points.



③ 2^{nd} derivative :

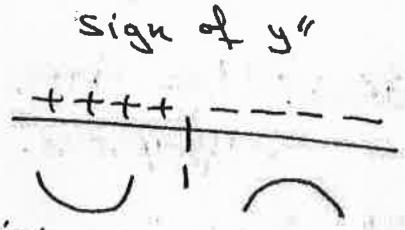
$$y' = 1 + \frac{3}{(x-1)^2}$$

$$y'' = 0 + \frac{[(x-1)^2 * 0] - [3 * 2(x-1)(1)]}{(x-1)4} = \frac{-6\cancel{(x-1)}}{(x-1)^{\cancel{4}\,3}}$$

$$\Rightarrow y'' = \frac{-6}{(x-1)^3}$$

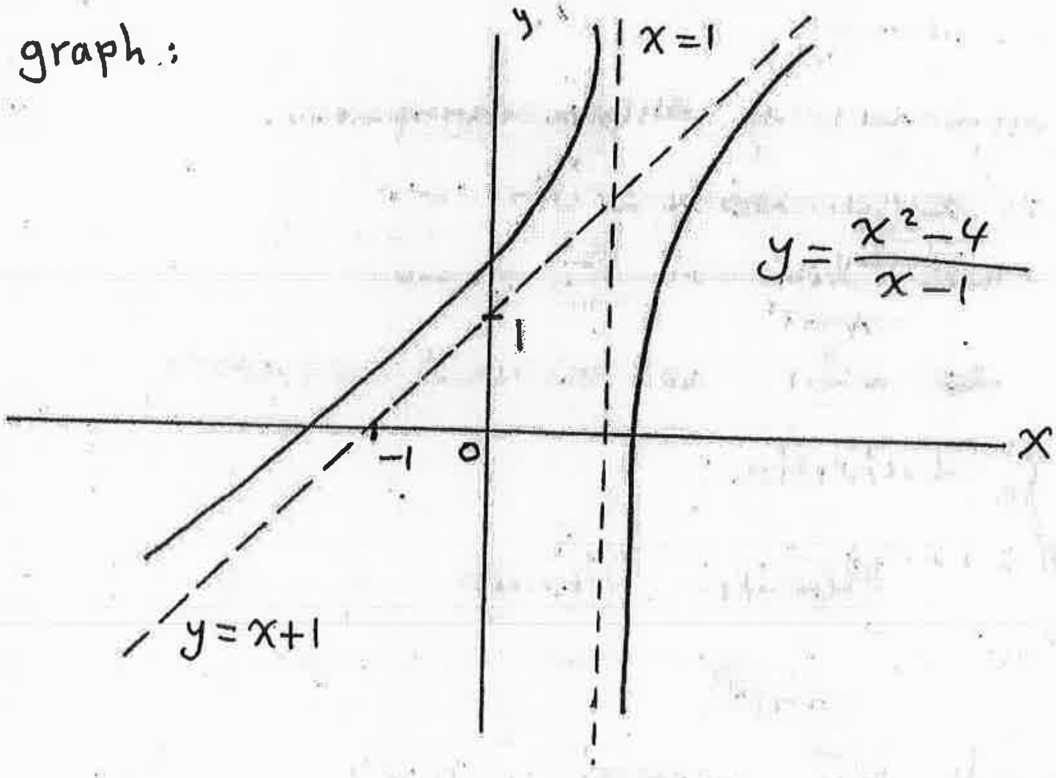
$$y'' = 0 \Rightarrow \frac{-6}{(x-1)^3} \neq 0$$

$x-1=0 \Rightarrow x=1$ (y'' not defined)



$x=1$ out of domain $\Rightarrow$ no inflection point.

④ The graph :



4.5 : Optimization :

Optimization means making the thing optimal.

The principles of maxima and minima are used to solve the optimization problems.

In these problems, we need the absolute maximum and the absolute minimum values to obtain the solutions.

Steps for solution :

1- Write an equation that relates the quantity that is to be maximized or minimized (call it now Q) with the variables which it depends on them (call them now x, y, z, ... etc.).

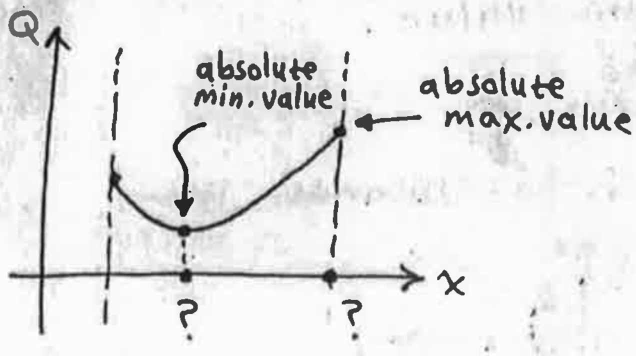
Or : $Q = f(x, y, z, \dots)$

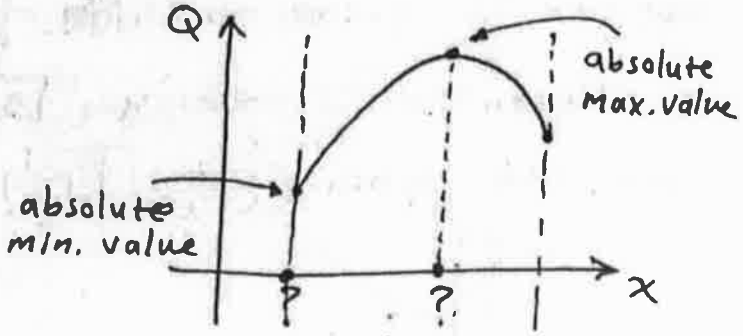
2- If Q has been expressed as a function of more than one variable, use the given information to find relationships among these variables. Thus Q will be expressed as a function of one variable. Write the domain of this function.

Or : $Q = f(x, y, z, \dots) \Rightarrow Q = f(x) \quad , a < x < b$

3- Differentiate Q with respect to the variable x.

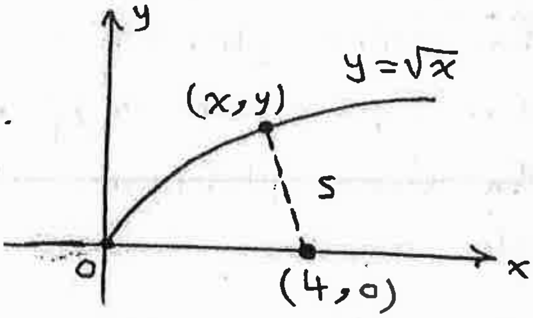
4- Find the value of x that makes Q to be absolute maximum or absolute minimum value. Also find the remaining variables y, z, ... etc.





Examples:

Ex.1: Find the coordinates for the point on the curve $y=\sqrt{x}$ that is nearest to the point $(4,0)$.



Sol. Let s = distance between the points (x,y) and $(4,0)$.

$$s=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$s=\sqrt{(x-4)^2+(y-0)^2}$$

$$s=\sqrt{x^2-8x+16+y^2}$$

but $y=\sqrt{x} \Rightarrow y^2=x \Rightarrow s=\sqrt{x^2-8x+16+x}$

$$\Rightarrow s=\sqrt{x^2-7x+16} \quad , \quad 0\leqslant x<\infty$$

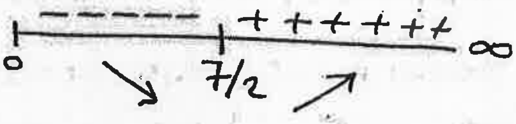
$$\frac{ds}{dx}=\frac{2x-7}{2\sqrt{x^2-7x+16}}$$

$$\frac{ds}{dx}=0 \Rightarrow \frac{2x-7}{2\sqrt{x^2-7x+16}}=0 \Rightarrow 2x-7=0 \Rightarrow x=\frac{7}{2}$$

Check the local minimum point:

From the sign of 1^{st} derivative,

$\Rightarrow x=\frac{7}{2}$ gives local min. value.



Check the bounds:

at $x=0 \Rightarrow s=\sqrt{16}=4$ units.

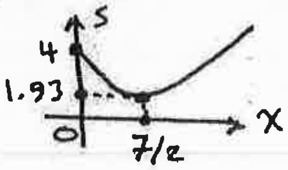
as $x\to\infty \Rightarrow s\to\infty$

Now: at $x=\frac{7}{2} \Rightarrow s=\sqrt{(\frac{7}{2})^2-7(\frac{7}{2})+16}=\frac{\sqrt{15}}{2}\approx 1.93$ units (o.k.)

$\Rightarrow x=\frac{7}{2}$ gives absolute min. value.

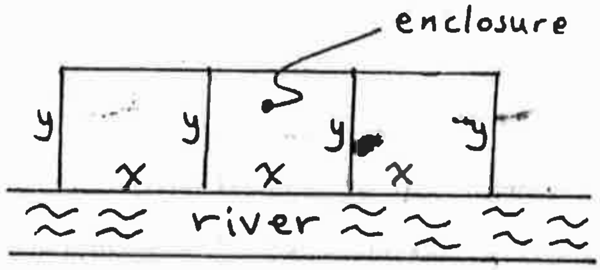
when $x=\frac{7}{2} \Rightarrow y=\sqrt{x}=\sqrt{\frac{7}{2}}$

$\Rightarrow$ the point $(\frac{7}{2},\sqrt{\frac{7}{2}})$ is the nearest point.



Ex. 2: A farmer wants to make three identical rectangular enclosures along a straight river. If he has 1200 m of fence, and if the sides along the river need no fence, what should be the dimensions of each enclosure if the total area is to be maximum.

sol. Let the dimensions for the enclosure are x & y.



and let the total area of enclosures = A

$$A = 3xy \quad \text{------} ①$$

The length of the fence $= 3x + 4y$

$$\Rightarrow \quad 1200 = 3x + 4y \Rightarrow 3x = 1200 - 4y$$

$$\Rightarrow x = 400 - \frac{4}{3}y \qquad \text{substitute into eq. ①}$$

$$\Rightarrow A = 3\left(400 - \frac{4}{3}y\right)y$$

$$\Rightarrow \quad A = 1200y - 4y^2 \qquad 0 < y \leqslant 300$$

$$\frac{dA}{dy} = 1200 - 8y$$

$$\frac{dA}{dy} = 0 \Rightarrow 1200 - 8y = 0 \Rightarrow y = \frac{1200}{8} = 150 \text{ m}$$

Check the local max. point :

$$\frac{d^2A}{dy^2} = -8 \quad \text{(negative)} \Rightarrow y = 150 \text{ gives max. value}$$

Check the bounds :

$$\text{as } y \to 0 \Rightarrow A \to 0$$

$$\text{as } y \to 300 \Rightarrow A \to 0$$

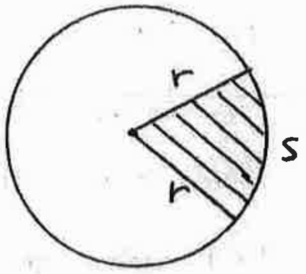
Now : at $y = 150 \Rightarrow A = 1200(150) - 4(150)^2 = 90{,}000\,m^2$ (o.k.)

$\Rightarrow y = 150$ gives absolute max. value.

When $y = 150 \Rightarrow x = 400 - \frac{4}{3}(150) = 200 \Rightarrow \boxed{\begin{array}{l} x = 200 \text{ m} \\ y = 150 \text{ m} \end{array}}$

Ex.3 : If the perimeter of the circular sector shown is 100 ft, what values of r and s will give the sector the greatest area?

Sol. Let area of sector = A
and perimeter = P



$$A = \frac{1}{2} r s \quad\text{———} ①$$

but $P = 2r + s$

$$100 = 2r + s \Rightarrow s = 100 - 2r$$

Substitute into ① $\Rightarrow A = \frac{1}{2} r(100 - 2r)$

$$\Rightarrow A = 50r - r^2 \qquad 0 < r < \underset{\sim\sim}{50}$$

$$\frac{P}{2} = \frac{100}{2} = 50$$

$$\frac{dA}{dr} = 50 - 2r$$

$$\frac{dA}{dr} = 0 \Rightarrow 50 - 2r = 0 \Rightarrow r = 25 \text{ ft}$$

Check the local max. point :

$$\frac{d^2A}{dr^2} = -2 \quad \text{(negative)} \Rightarrow r = 25 \text{ ft gives max. value}$$

Check the bounds :

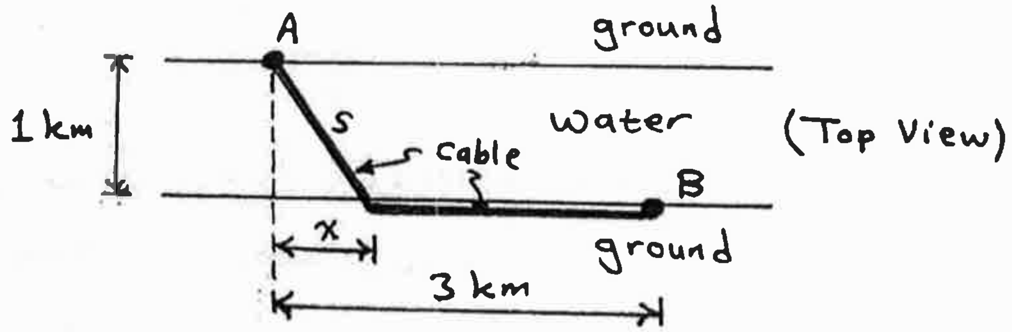
as $r \to 0 \Rightarrow A \to 0$
as $r \to 50 \Rightarrow A \to 0$

Now at $r = 25 \Rightarrow A = 50(25) - (25)^2 = 625 \text{ ft}^2$ (o.k.)

at $r = 25 \Rightarrow s = 100 - 2(25) = 50$ ft

$$\Longrightarrow \boxed{r = 25 \text{ ft}} \text{ and } \boxed{s = 50 \text{ ft}}$$ give greatest area for the sector.

Ex.4 : The cost to lay a cable underwater is 5 million Dinars per kilometer, and to lay it underground is 3 million Dinars per kilometer. Find the distance x which minimize the cost of connecting the cable from A to B. (See the figure below).



sol.

Let the cost of connecting $= C$

$$S = \sqrt{1^2 + x^2} = \sqrt{1+x^2} \qquad [\text{Pythagorean theorem}]$$

$$\Rightarrow C = \sqrt{1+x^2} * 5 + (3-x) * 3$$

$$C = 5 * \sqrt{1+x^2} - 3x + 9 \qquad 0 \leqslant x \leqslant 3$$

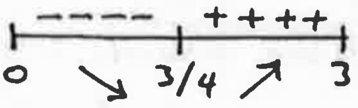
$$\frac{dC}{dx} = 5 * \frac{2x}{2\sqrt{1+x^2}} - 3 = \frac{5x}{\sqrt{1+x^2}} - 3$$

$$\frac{dC}{dx} = 0 \Longrightarrow \frac{5x}{\sqrt{1+x^2}} = 3 \Longrightarrow 3\sqrt{1+x^2} = 5x$$

squaring both sides $\Longrightarrow 9(1+x^2) = 25x^2 \Longrightarrow 9 + 9x^2 = 25x^2$

$$\Rightarrow 16x^2 = 9 \Rightarrow x^2 = \frac{9}{16} \Longrightarrow x = \frac{3}{4} \text{ km}$$

Check the local min. value :



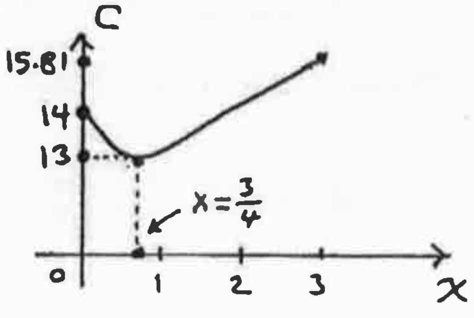
$\Rightarrow x = \frac{3}{4}$ gives local min. value.

at $x = \frac{3}{4} \Longrightarrow C = 13$ million Dinars.

Check the bounds :

at $x = 0 \Longrightarrow C = 14$ million Dinars.

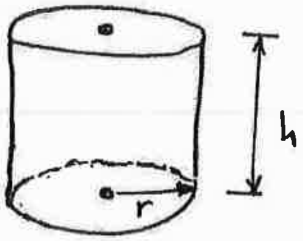
at $x = 3 \Longrightarrow C = 15.81$ million Dinars.



$\Longrightarrow \boxed{x = \frac{3}{4} \text{ km}}$ gives minimum cost.

Ex.5: A cylindrical can is to be made to contain 100 in^3 of liquid. It's closed on the top and bottom. Find the dimensions that minimize the metal used.

Sol. Volume of cylindrical can = V



$$V = \pi r^2 h$$

$$100 = \pi r^2 h \Rightarrow h = \frac{100}{\pi r^2}$$

Area of metal used = Surface area of can = A

A = Area of top and bottom + Area of cylindrical surface

$$\Rightarrow A = 2\pi r^2 + 2\pi r h \longrightarrow ①$$

substitute $h = \frac{100}{\pi r^2}$ into eq. ① :

$$\Rightarrow A = 2\pi r^2 + 2\cancel{\pi}\cancel{r}\left(\frac{100}{\cancel{\pi} r^{\cancel{2}}}\right) = 2\pi r^2 + \frac{200}{r}$$

$$\Rightarrow A = 2\pi r^2 + \frac{200}{r} \qquad 0 < r < \infty$$

$$\frac{dA}{dr} = 4\pi r - \frac{200}{r^2}$$

$$\frac{dA}{dr} = 0 \Rightarrow 4\pi r = \frac{200}{r^2} \Rightarrow r^3 = \frac{50}{\pi} \Rightarrow r = \sqrt[3]{\frac{50}{\pi}} \text{ in.}$$

Check the min. value :

$$\frac{d^2A}{dr^2} = 4\pi + \frac{400}{r^3} \qquad +\text{ve in } (0, \infty) \Rightarrow \text{min. value.}$$

$$\Rightarrow \boxed{r = \sqrt[3]{\frac{50}{\pi}}} \text{ gives min. value.}$$

Check the bounds :

$$r \to 0 \Rightarrow A \to \infty$$

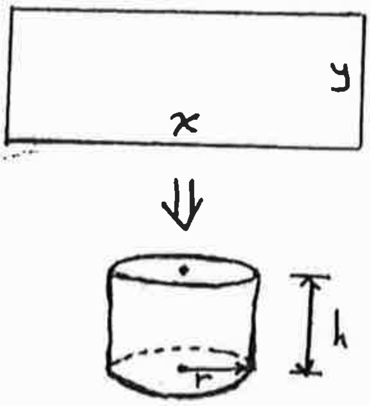
$$r \to \infty \Rightarrow A \to \infty$$

$$\text{Now: at } r = \sqrt[3]{\frac{50}{\pi}} \Rightarrow h = \frac{100}{\pi\left(\frac{50}{\pi}\right)^{2/3}} = \frac{\cancel{100}^{\,2}}{\cancel{\pi}\left(\frac{\cancel{50}}{\cancel{\pi}}\right)\left(\frac{50}{\pi}\right)^{-1/3}}$$

$$\Rightarrow h = 2\sqrt[3]{\frac{50}{\pi}} \quad \text{or} \quad \boxed{h = 2r}.$$

Ex. 6: A rectangle of perimeter 12 ft is to be used to form a cylinder by revolving it about its edges. Find the dimensions of this rectangle which will result a maximum volume of the cylinder.

Sol.



For the cylinder:

$h = y$

$2\pi r = x \Rightarrow r = \frac{x}{2\pi}$

Let volume of cylinder = V

$V = \pi r^2 h = \pi \left(\frac{x}{2\pi}\right)^2 \cdot y$

$V = \frac{1}{4\pi} x^2 y$ ——— ①

Let perimeter for rectangle = P

$P = 2x + 2y$

$12 = 2x + 2y \Rightarrow 6 = x + y \Rightarrow y = 6 - x$

Substitute into eq. ① $\Rightarrow$ $V = \frac{1}{4\pi} x^2 (6 - x) = \frac{1}{4\pi}(6x^2 - x^3)$

$\Rightarrow V = \frac{1}{4\pi}(6x^2 - x^3) \qquad 0 < x < 6$

$\frac{dV}{dx} = \frac{1}{4\pi}(12x - 3x^2)$

$\frac{dV}{dx} = 0 \Rightarrow \frac{1}{4\pi}(12x - 3x^2) = 0 \Rightarrow 12x - 3x^2 = 0$

$\Rightarrow -3(x^2 - 4x) = 0 \Rightarrow x^2 - 4x = 0 \Rightarrow x(x-4) = 0$

$\Rightarrow x = 0$ (neglected), and $x = 4$.

Check the max. value:

$\frac{d^2V}{dx^2} = \frac{1}{4\pi}(12 - 6x) \Rightarrow$ at $x = 4$, $\frac{d^2V}{dx^2} = -ve$

$\Rightarrow x = 4$ ft gives max. value.

Check the bounds:

as $x \to 0 \Rightarrow V \to 0$ and as $x \to 6 \Rightarrow V \to 0$

Now: at $x = 4 \Rightarrow y = 6 - 4 = 2 \Rightarrow V = \frac{8}{\pi} \approx 2.546$ ft^3 o.k.

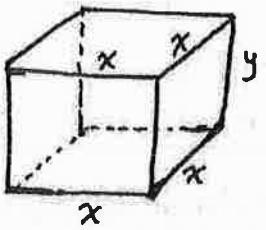
$\Rightarrow$ $\boxed{x = 4 \text{ ft}}$ and $\boxed{y = 2 \text{ ft}}$ give maximum volume.

Ex.7: A box with square base and top is to be made to contain 1250 ft^3. The material for the base costs 3500 Dinars per square foot, for the top costs 1500 Dinars per square foot, and for the sides costs 2000 Dinars per square foot. Find the dimensions that will minimize the cost of the box.

Sol.: Let the cost = C

$$C = 3500(x^2) + 1500(x^2) + 2000(4xy)$$

$$\Rightarrow C = 5000x^2 + 8000xy \quad \text{———} ①$$



Volume of the box = V

$$V = x^2y$$

$$1250 = x^2y \Rightarrow y = \frac{1250}{x^2}$$

substitute into eq. ① : $\Rightarrow C = 5000x^2 + 8000x\left(\frac{1250}{x^2}\right)$

$$\Rightarrow C = 5000x^2 + \frac{10^7}{x} \qquad 0 < x < \infty$$

$$\frac{dC}{dx} = 10^4 x - \frac{10^7}{x^2}$$

$$\frac{dc}{dx} = 0 \Rightarrow 10^4 x - \frac{10^7}{x^2} = 0 \Rightarrow \frac{10^4x^3 - 10^7}{x^2} = 0$$

$$\Rightarrow 10^4x^3 - 10^7 = 0 \Rightarrow x^3 = \frac{10^7}{10^4} = 1000 \Rightarrow x = 10$$

Check the minimum value :

$$\frac{d^2c}{dx^2} = 10^4 + \frac{2 * 10^7}{x^3} \qquad +ve \text{ in } (0,\infty) \Rightarrow \text{min. value.}$$

Check the bounds :

$x \to 0 \Rightarrow C \to \infty$

$x \to \infty \Rightarrow C \to \infty$

Now : at $x = 10 \Rightarrow C = 1{,}500{,}000$ Dinars. O.k.

at $x = 10 \Rightarrow y = \frac{1250}{(10)^2} = 12.5$

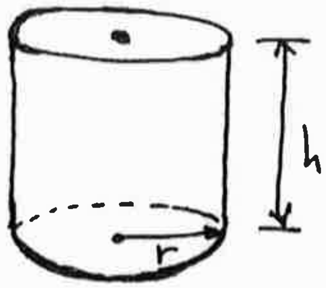
$\Rightarrow \boxed{x = 10 \text{ ft}}$ and $\boxed{y = 12.5 \text{ ft}}$ give min. cost.

x.8 : A cylindrical open pool is to be made to contain 5 m^3 of water. The material for the bottom costs twice as much as the material for the sides. Find the dimensions of the pool to minimize the cost of construction.

Sol. : Let the total cost = C

the cost of 1 m^2 of the material of the bottom = C_b

the cost of 1 m^2 of the material of the sides = C_s



$$\Rightarrow \quad C_b = 2\,C_s$$

The total cost is : $C = 2\pi r h\,(C_s) + \pi r^2 (C_b)$

$$\Rightarrow \quad C = 2\pi r h\,(C_s) + \pi r^2\,(2\,C_s)$$

$$\Rightarrow \quad C = 2\,C_s\,(\pi r h + \pi r^2) \text{ ——— } ①$$

The volume of cylinder is : $V = \pi r^2 h$

$$\Rightarrow 5 = \pi r^2 h \Rightarrow h = \frac{5}{\pi r^2}$$

Substitute into eq. ①:

$$\Rightarrow \quad C = 2C_s\left[\pi r\left(\frac{5}{\pi r^2}\right) + \pi r^2\right]$$

$$C = 2C_s\left(\frac{5}{r} + \pi r^2\right) \qquad 0 < r < \infty$$

$$\frac{dC}{dr} = 2\,C_s\left(\frac{-5}{r^2} + 2\pi r\right)$$

$$\frac{dC}{dr} = 0 \Longrightarrow 2C_s\left(\frac{-5}{r^2} + 2\pi r\right) = 0 \Rightarrow \frac{-5}{r^2} + 2\pi r = 0$$

$$\Longrightarrow \frac{5}{r^2} = 2\pi r \Rightarrow r^3 = \frac{5}{2\pi} \Rightarrow r = \sqrt[3]{\frac{5}{2\pi}}$$

Check for min. value : $\frac{d^2C}{dr^2} = 2\,C_s\left(\frac{10}{r^3} + 2\pi\right) \rightarrow +ve \Rightarrow$ min. value

Check the bounds : as $r \rightarrow 0 \Rightarrow C \rightarrow \infty$ and as $r \rightarrow \infty \Rightarrow C \rightarrow \infty$

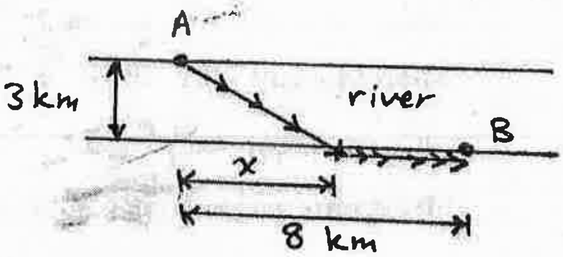
Now : at $r = \sqrt[3]{\frac{5}{2\pi}} \Rightarrow h = \dfrac{5}{\pi\left(\frac{5}{2\pi}\right)^{2/3}} = 2 * \sqrt[3]{\frac{5}{2\pi}} = 2r$

$\Longrightarrow \boxed{r = \sqrt[3]{\frac{5}{2\pi}} \text{ m}}$ and $\boxed{h = 2r}$ give minimum cost.

PROBLEMS :

① A man launches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible. If he can row 6 km/hr and run 8 km/hr, where should he land to reach B as soon as possible?

Ans. $x = \frac{9}{\sqrt{7}} \doteq 3.4$ km



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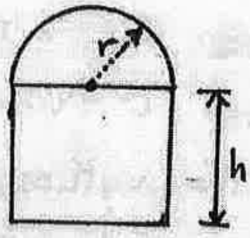
② The window shown in the figure below is made from a given frame length (L). Find the ratio (h/r) that maximize the area of this window.

Ans. $\frac{h}{r} = 2$



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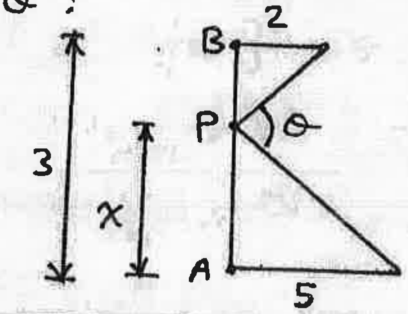
③ Where should the point P be chosen on the line segment AB so as ~~to~~ maximize the angle θ?

Ans. $x = 5 - 2\sqrt{5}$



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④ A wire 28 m long is to be cut into two pieces, one piece to form a square and the other to form a circle. How should the wire be cut so as to:

(a) Maximize the sum of two areas.
(b) Minimize the sum of two areas.

Ans. (a) all wire used for circle.
(b) 12.31 m for circle and 15.69 m for square.

CHAPTER FIVE

INTEGRATION

Calculus consists of two main branches, the first branch is the differential calculus, and the second branch is the integral calculus.

This chapter introduces the integral calculus.

Integration is the process of calculating an integral.

Antiderivatives:

The antiderivative is the function obtained from its derivative. It is denoted by F.

For example:

$f(x) = \sin x \Rightarrow f'(x) = \cos x \Rightarrow F(x) = \sin x$

$$\begin{tikzcd}
\boxed{\sin x} \arrow[r, shift left, "\text{derivative}"] & \boxed{\cos x} \arrow[l, shift left, "\text{antiderivative}"]
\end{tikzcd}$$

To find the antiderivative of a function we depends on two corollaries:

① If $f'(x)=0$ then $F(x)=C$ [c: constant]

Ex. $f'(x)=0 \Rightarrow F(x)=3$ (or any constant)

② If $f_1'(x)=f_2'(x) \Rightarrow F_1(x)=F_2(x)+C$

Ex. $f_1(x)= x^2+3 \Rightarrow f_1'(x)=2x$

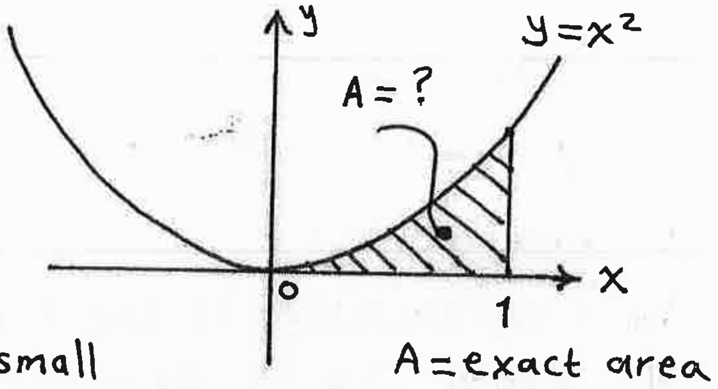
$f_2(x)= x^2-5 \Rightarrow f_2'(x)=2x$

$f_1'(x)=f_2'(x) \Rightarrow F_1(x)=F_2(x)+C$

The area problem :

For example, if we have the function $y=x^2$, and we want to find the area under the graph from $x=0$ to $x=1$.

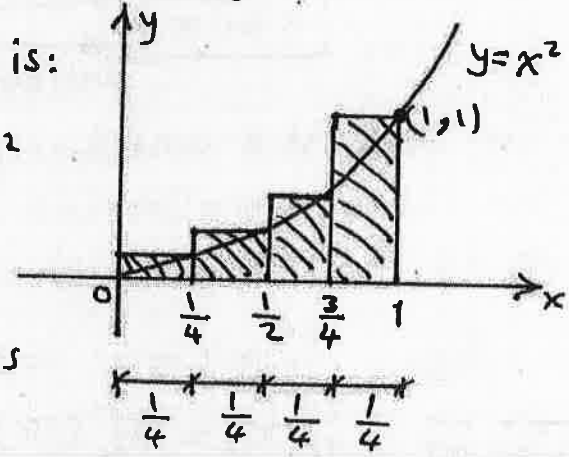
It is not easy to find the area of a region with curved sides.



But we can estimate this area by dividing it into small strips.

If the shaded region is divided into four strips of rectangular shapes with base equals $\frac{1}{4}$ unit and height equals to the right edge of the rectangle.

$\Rightarrow$ the area of these rectangles is:



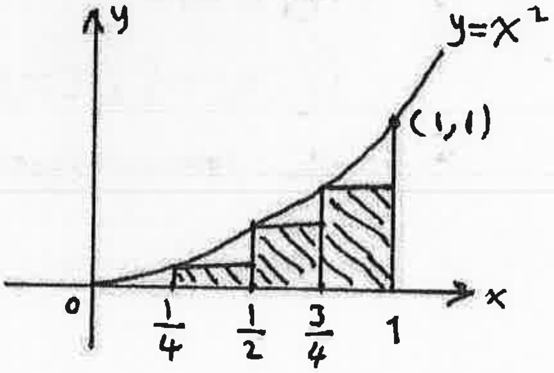
$$R_4 = \frac{1}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\left(\frac{3}{4}\right)^2 + \frac{1}{4}(1)^2$$

$$= \frac{15}{32} = 0.46875$$

Notice that the exact area is less than R_4. Or $A < 0.46875$

Now, if the area is divided into four rectangles with height equals to the left edge of the rectangle.

$\Rightarrow$ The area of these rectangles is:



$$L_4 = \frac{1}{4}(0)^2 + \frac{1}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\left(\frac{3}{4}\right)^2$$

$$= \frac{7}{32} = 0.21875$$

Notice that $A > 0.21875$

We see that: $0.21875 < A < 0.46875$

Now, if we repeat this procedure with a larger number of rectangular strips we obtain the following results :

n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

n: number of rectangles.

Notice that for $n = 1000$: $L_n < A < R_n$

the average of L_n and $R_n \approx 0.3333335$

this value closed to exact area $A = \frac{1}{3} \approx 0.3333333$ obtained from integration.

From this example, we can see that the increase of the number of strips makes the area result more accurate.

To make the calculated area to be exact, we need :

① Formula for sums of large numbers of terms.

② Finding the limit of such sums as the number of terms tends to infinity.

① Formula for sums :

The Greek capital letter $\sum$ (sigma) is used to indicate sums.

For example,

the sum $(a_1+a_2+a_3+\cdots+a_n)$ is written in sigma notation as :

$\sum_{k=1}^{n} a_k$ (the sum of a sub k from a=1 to a=n).

in which : k : index of summation = integers from 1 to n.

a_k : k^{th} term.

a_1 : first term.

a_n : last term (n^{th} term).

1 : lower limit of summation.

n : upper limit of summation.

Ex.1: Find $\sum_{k=0}^{2} \frac{1}{2^k}$:

sol. $\sum_{k=0}^{2} \frac{1}{2^k} = \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} = \frac{1}{1} + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$

Ex.2: Find $\sum_{k=-3}^{-1} (k+1)$.

sol. $\sum_{k=-3}^{-1} (k+1) = (-3+1) + (-2+1) + (-1+1) = -2-1+0 = -3$

② Finding the limit of of sums :

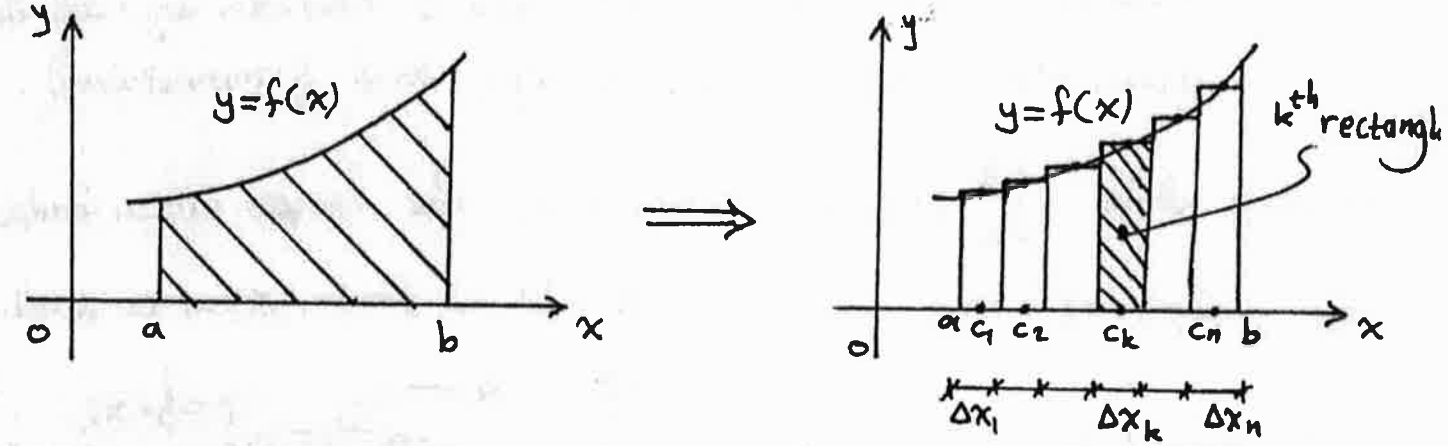
The limit of sums as the number of terms approaches infinity can be written as :

$\lim_{n\to\infty} \sum_{k=1}^{n} a_k$.

Definite Integral :

We have the function $y=f(x)$, and we want to find the exact area under the graph of this function from $x=a$ to $x=b$.

To find this area, divide it into n rectangles :

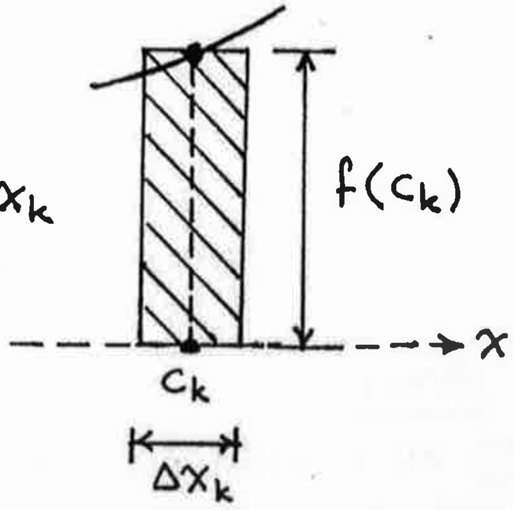


Take a typical rectangle
(k^{th} rectangle) :

The area of k^{th} rectangle $= f(c_k) \cdot \Delta x_k$

If the sum of areas of rectangles is denoted by S, then :

$$S = \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$$



Now, if the exact area under the curve of $y=f(x)$ from $x=a$ to $x=b$ is denoted by A, then :

$$A = \lim_{n \to \infty} \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$$

The definite integral of $f(x)$ from $x=a$ to $x=b$ is :

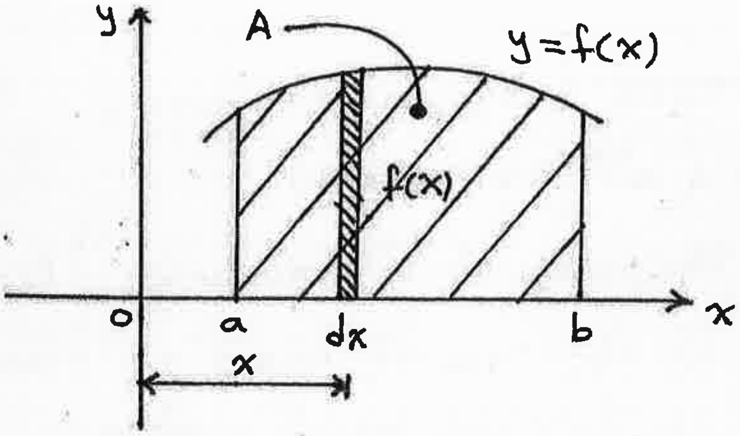
$$\boxed{\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k}$$

Where :

a : Lower limit of integration.
b : Upper limit of integration.
f(x) : Integrand of the integral.
dx : Differential (index of integration).
$\int$: Integral sign (it is elongated S chosen by Leibniz from the letter s in German word summation).

Notice that $\int_a^b f(x)\,dx$ represents the exact area under the graph of the function $y=f(x)$ from $x=a$ to $x=b$.

$$A = \text{exact area} = \int_a^b f(x)\,dx$$



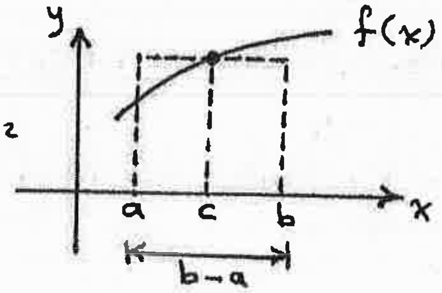
Notes:

① All continuous functions are integrable.

② If f(x) is negative ; the area becomes below the x-axis and then the area becomes negative number.

Definition : The average value of f(x) on [a,b] is :

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx .$$



For example: The average value of $y=x^2$ from $x=0$ to $x=1$ is :

$$= \frac{1}{1-0}\left(\frac{1}{3}\right) = \frac{1}{3}$$

Rules for definite integral:

① $\int_a^a f(x)\,dx = 0$

② $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

③ $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$

④ $\int_a^b [f(x) \mp g(x)]\,dx = \int_a^b f(x)\,dx \mp \int_a^b g(x)\,dx$

⑤ $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

⑥ $\int_b^c f(x)\,dx = \int_a^c f(x)\,dx - \int_a^b f(x)\,dx$

⑦ $g(x) \geqslant f(x) \Rightarrow \int_a^b g(x)\,dx \geqslant \int_a^b f(x)\,d$

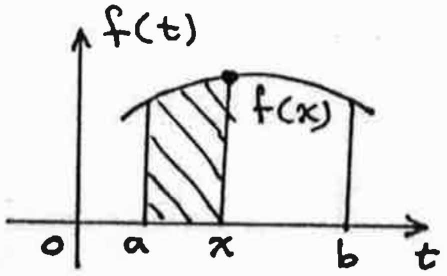
⑧ $f(x) \geqslant 0 \Rightarrow \int_a^b f(x)\,dx \geqslant 0$

The Fundamental Theorems of Integral Calculus:

There are two fundamental theorems of integral calculus.

The first fundamental theorem:

If f is continuous on [a, b], then the function $F(x) = \int_a^x f(t)\,dt$ has a derivative at every point on [a, b] and:



$$\boxed{\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)}$$

Ex.1: Find dy/dx for $y = \int_{-\pi}^{x} \cos t\,dt$.

Sol.: $\frac{dy}{dx} = \frac{d}{dx}\int_{-\pi}^{x} \cos t\,dt = \cos x$

Ex.2: Find dy/dx for $y = \int_1^{x^2} \cos t\,dt$.

Sol.: Let $u = x^2 \Rightarrow y = \int_1^u \cos t\,dt$.

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \cos u \cdot \frac{du}{dx} = \cos x^2 \cdot 2x = 2x\cos x^2$$

The second fundamental theorem (the integral evaluation theorem):

If f is continuous at every point on [a, b] and F is any antiderivative of f on [a, b] then:

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

Ex.: $\int_0^{\pi} \cos x\,dx = [\sin x]_0^{\pi} = \sin \pi - \sin 0 = 0 - 0 = 0$

Indefinite Integrals:

If the function f(x) is a derivative, then the set of all antiderivatives of f is called the indefinite integral of f.

The form of the definite integral is: $\int f(x)\,dx$.

The value of this integral is $F(x) + C$.

Where, $F(x)$: Antiderivative

C : Constant of integration (or arbitrary constant).

Therefore: $$\boxed{\int f(x)\,dx = F(x) + C}$$

Integration Formulas: If $u = f(x)$:

① $\int u^n\,du = \frac{u^{n+1}}{n+1} + C$

② $\int \sin u\,du = -\cos u + C$

③ $\int \cos u\,du = \sin u + c$

④ $\int \sec^2 u\,du = \tan u + c$

⑤ $\int \sec u \tan u\,du = \sec u + c$

⑥ $\int \csc^2 u\,du = -\cot u + c$

⑦ $\int \csc u \cot u\,du = -\csc u + C$

Rules:

① $\int \frac{dF}{dx}\,dx = \int dF = F(x) + C$

② $\frac{d}{dx}\int f(x)\,dx = f(x)$

Examples:

① $\int x^3\,dx = \frac{x^4}{4} + C$ $\qquad$ $\left[\text{ from } \int u^n = \frac{u^{n+1}}{n+1} + C\right]$

② $\int 3\sqrt{x}\,dx = 3\int x^{1/2}\,dx = 3 * \frac{x^{3/2}}{3/2} + C = 2x^{3/2} + C$

③ $\int(\sec x\tan x - \sec^2 x)\,dx = \int \sec x\tan x\,dx - \int\sec^2 x\,dx$

$= \sec x - \tan x + C$

④ $\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \int \frac{1}{2}\,dx + \int \frac{1}{2}\cos 2x\,dx$

Now: $\int \frac{1}{2}\cos 2x\,dx = \frac{1}{2}\int \cos 2x\,dx$

compare $\int \cos 2x\,dx$ with the formula $\int \cos u\,du$

$\Rightarrow$ Let $u = 2x \Rightarrow \frac{du}{dx} = 2 \quad \Rightarrow du = 2dx \Rightarrow dx = \frac{du}{2}$

therefore $\int \cos 2x\,dx \longrightarrow \int \cos u \cdot \frac{du}{2} = \frac{1}{2}\int \cos u . du$

$\Rightarrow \frac{1}{2}\int \cos u\,du = \frac{1}{2}\sin u \quad = \quad \frac{1}{2}\sin 2x$

Return to original integral:

$\int \frac{1}{2}\,dx + \int \frac{1}{2}\cos 2x\,dx = \frac{1}{2}(x) + \frac{1}{2}\left(\frac{1}{2}\sin 2x\right) + C$

$= \frac{x}{2} + \frac{1}{4}\sin 2x + C$

⑤ $\int(1+\tan^2\theta)\,d\theta = \int \sec^2\theta\,d\theta = \tan\theta + C$

Integration by Substitution :

The integration by substitution is used to simplify the evaluation of integral.

If we have the integral: $\int f(g(x)) \cdot g'(x).dx$.

substitute $u = g(x) \Rightarrow du = g'(x)\, dx$

to obtain $\int f(u)\, du$.

Ex.1: Evaluate $\int \sqrt{1+x^2} \cdot x\,dx$

sol.: Let $u = 1+x^2 \Rightarrow du = 2x\,dx \Rightarrow x\,dx = \frac{du}{2}$

$$\longrightarrow \int \sqrt{u} \cdot \frac{du}{2} = \frac{1}{2}\int u^{1/2}\, du = \frac{1}{2}\left(\frac{u^{3/2}}{\frac{3}{2}}\right) + C$$

$$= \frac{1}{3} u^{3/2} + c = \frac{1}{3}(1+x^2)^{3/2} + C$$

Ex.2: Evaluate $\int \cos(7x+5)\, dx$

sol.: Let $u = 7x+5 \Rightarrow du = 7dx \Rightarrow dx = \frac{du}{7}$

$$\longrightarrow \int \cos u \cdot \frac{du}{7} = \frac{1}{7}\int \cos u\, du = \frac{1}{7} \sin u + c.$$

$$= \frac{1}{7} \sin(7x+5) + C$$

Ex.3: Evaluate $\int \frac{\cos 2x}{\sqrt{\sin 2x + 3}}\, dx$

sol. Let $u = \sin 2x + 3 \Rightarrow du = 2\cos 2x\, dx$

$\Rightarrow \cos 2x\, dx = \frac{du}{2}$

$$\longrightarrow \int u^{-1/2} \cdot \frac{du}{2} = \frac{1}{2}\int u^{-1/2}\, du = \frac{1}{2}\, \frac{u^{1/2}}{1/2} + c$$

$$= u^{1/2} + C = \sqrt{\sin 2x + 3} + C$$

Ex.4 : Evaluate $\int \frac{x \cos\sqrt{3x^2-6}}{\sqrt{3x^2-6}}\,dx$

sol. Let $u = \sqrt{3x^2-6}$

$$\frac{du}{dx} = \frac{1}{2\sqrt{3x^2-6}} * 6x$$

$$\Rightarrow du = \frac{3x\,dx}{\sqrt{3x^2-6}} \Longrightarrow \frac{x\,dx}{\sqrt{3x^2-6}} = \frac{du}{3}$$

$$\longrightarrow \int \cos u \cdot \frac{du}{3} = \frac{1}{3}\int \cos u\,du = \frac{1}{3}\sin u + c$$

$$= \frac{1}{3}\sin\sqrt{3x^2-6} + C$$

Ex.5 (Substitution in definite integral):

Evaluate $\int_0^{\pi/4} \tan x \sec^2 x\,dx$

sol. Let $u = \tan x \Rightarrow du = \sec^2 x\,dx$

$$\left.\begin{array}{l}\text{Lower limit} = \tan 0 = 0 \\ \text{Upper limit} = \tan\frac{\pi}{4} = 1\end{array}\right\} \text{from } u = \tan x$$

$$\longrightarrow \int_0^1 u\,du = \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2}\cdot[u^2]_0^1 = \frac{1}{2}[1^2 - 0^2]$$

$$= \frac{1}{2}(1) = \frac{1}{2}$$

Ex.6: Evaluate $\int_{\frac{\pi^2}{4}}^{\pi^2} \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

sol.: $u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}dx \Rightarrow \frac{dx}{\sqrt{x}} = 2du$

U.L. $= \sqrt{\pi^2} = \pi$

L.L. $= \sqrt{\frac{\pi^2}{4}} = \frac{\pi}{2}$

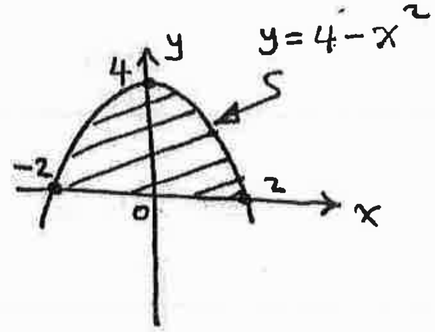
$$\longrightarrow \int_{\frac{\pi}{2}}^{\pi} \sin u \cdot 2du = 2\int_{\frac{\pi}{2}}^{\pi} \sin u\,du = -2\left[\cos u\right]_{\frac{\pi}{2}}^{\pi} = -2\left[\cos\pi - \cos\frac{\pi}{2}\right]$$

$$= -2(-1-0) = 2$$

Examples about the areas:

Ex.1: Find the area between the x-axis and the curves of ⓐ $y=4-x^2$ ⓑ $y=x^2-4$ for $-2 \leqslant x \leqslant 2$.

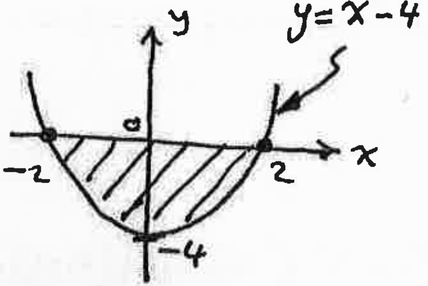
Sol. ⓐ Area under curve $= \int_a^b f(x)\,dx$



$$\Rightarrow \text{Area} = \int_{-2}^{2} (4-x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$

$$= \left(4(2) - \frac{2^3}{3}\right) - \left(4(-2) - \frac{(-2)^3}{3}\right)$$

$$= \frac{32}{3} \text{ units}$$



ⓑ $\text{Area} = \int_{-2}^{2} (x^2-4)\,dx$

$$= \left[\frac{x^3}{3} - 4x\right]_{-2}^{2} = -\frac{32}{3} \text{ units}$$

the minus sign means that the area is below x-axis.

Note: When the graph has both +ve and −ve values of area, solve as the following steps:

1- Find the points where $f=0$.

2- Use the zero values to partition $[a,b]$ into subintervals.

3- Integrate f over each subinterval.

4- Add the absolute values of results.

Ex.2: Find the area of the region between the x-axis and the curve $y=x^3-4x$, $-2 \leqslant x \leqslant 2$.

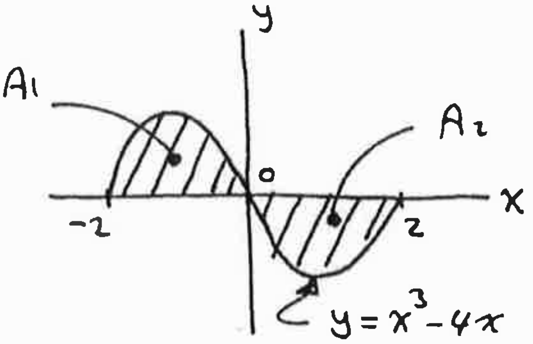
Sol. $y=0 \Rightarrow x^3-4x=0 \Rightarrow x(x^2-4)=0$

$\Rightarrow x(x-2)(x+2)=0 \Rightarrow x=0$, $x=2$, and $x=-2$.

the subintervals are $[-2,0]$ and $[0,2]$

$$A_1 = \int_{-2}^{0} (x^3 - 4x)\, dx$$

$$= \left[\frac{x^4}{4} - 2x^2\right]_{-2}^{0} = [0] - [4-8] = 4$$



$$A_2 = \int_{0}^{2} (x^3 - 4x)\, dx = \left[\frac{x^4}{4} - 2x^2\right]_{0}^{2}$$

$$= [4-8] - [0] = -4$$

$\Rightarrow$ Total area $= |A_1| + |A_2| = |4| + |-4| = 4 + 4 = 8$ units

---

Solving the initial value problems:

Given $\frac{dy}{dx} = f(x)$, this equation called differential equation.

To solve this equation: $dy = f(x)\, dx \Rightarrow \int dy = \int f(x) dx$

$\Rightarrow y = F(x) + C$ this is general solution.

To find the particular solution, use the initial conditions $y = y_0$ and $x = x_0$ to find C.

Ex.1 Solve the following differential equation (or solve the following initial value problem):

$$\frac{dy}{dx} = 3x^2 - 2x - 1$$, the initial conditions: $y = 10$ at $x = 1$

sol. $dy = (3x^2 - 2x - 1)\, dx \Rightarrow \int dy = \int (3x^2 - 2x - 1) dx$

$\Rightarrow y = x^3 - x^2 - x + C$ general solution

at $x = 1$, $y = 10$:

substitute $\Rightarrow$ $10 = (1)^3 - (1)^2 - 1 + C \Rightarrow C = 11$

$\Rightarrow y = x^3 - x^2 - x + 11$ particular solution.

Ex.2: Derive the standard equation for free fall $S(t)$ near the surface of every planet.

Hint: Differential equation $\frac{d^2s}{dt^2} = g$

Initial conditions are:

$\frac{ds}{dt} = V_o$ and $S = S_o$ at $t = 0$.

sol. $$\frac{d^2s}{dt^2} = g$$

integrate with dt:

$$\Rightarrow \int \frac{d^2s}{dt^2}\,dt = \int g\,dt$$

$$\frac{ds}{dt} = gt + C_1$$

at $t=0$, $\frac{ds}{dt} = V_o \Rightarrow V_o = g(0) + C_1 \Rightarrow C_1 = V_o$

$$\Rightarrow \frac{ds}{dt} = gt + V_o$$

integrate with dt again (or by: $\int ds = \int (gt+V_o)dt$)

$$\Rightarrow \int \frac{ds}{dt}\,dt = \int (gt + V_o)\,dt$$

$$s = g\frac{t^2}{2} + V_o(t) + C_2$$

at $t=0$, $S = S_o \Rightarrow S_o = g\left(\frac{0}{2}\right) + V_o(0) + C_2$

$$\Rightarrow C_2 = S_o$$

$$\Rightarrow \boxed{S = \frac{1}{2}gt^2 + V_o t + S_o}$$ *standard equation.*

Numerical Integration :

When we cannot compute the value of an integral exactly, we approximate this value numerically. The numerical methods of integration are specially useful for approximating integral of functions that are available only in graphical or tabular form.

We will study three methods. These methods are:

1- Trapezoidal method.

2- Midpoint method.

3- Simpson's rule method.

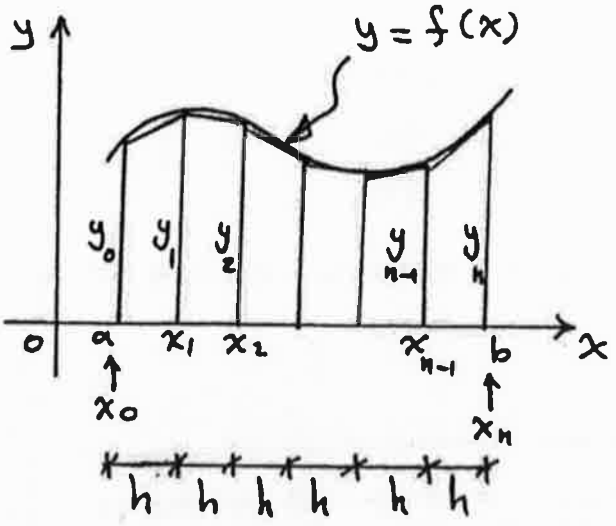
---

① Trapezoidal method :

We want to estimate $\int_a^b f(x)\,dx$.

In this method, the area under the curve is divided into a number of trapezoids.

The value of the integral $\int_a^b f(x)\,dx$ is estimated by the sum of areas of these trapezoids.



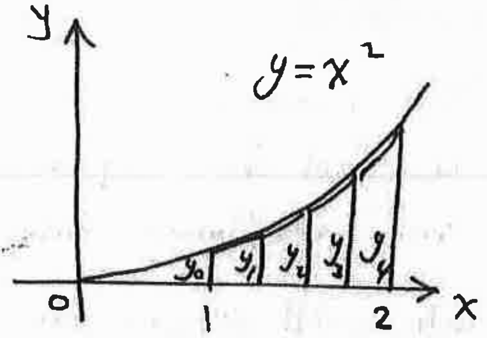
$$\Longrightarrow \boxed{T = \frac{h}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots\cdots + 2y_{n-1} + y_n\right)}$$

Where : T = approximated value of integral by trapezoids.
h = width of trapezoid = $\frac{b-a}{n}$
$y_0 \longrightarrow y_n$ = Values of the function $f(x)$ at $x_0 \longrightarrow x_n$.
n = number of trapezoids.

Ex. Use the trapezoidal method with $n=4$ to estimate the integral $\int_1^2 x^2\,dx$.

Sol. $n=4$, $a=1$, $b=2$

$$h=\frac{b-a}{n}=\frac{2-1}{4}=\frac{1}{4}$$



Make a table :

x	$y=x^2$
$x_0=1$	$y_0=(1)^2=1$
$x_1=1+\frac{1}{4}=\frac{5}{4}$	$y_1=(\frac{5}{4})^2=\frac{25}{16}$
$x_2=\frac{5}{4}+\frac{1}{4}=\frac{6}{4}$	$y_2=(\frac{6}{4})^2=\frac{36}{16}$
$x_3=\frac{6}{4}+\frac{1}{4}=\frac{7}{4}$	$y_3=(\frac{7}{4})^2=\frac{49}{16}$
$x_4=\frac{7}{4}+\frac{1}{4}=2$	$y_4=(2)^2=4$

$$T=\frac{h}{2}\left(y_0+2y_1+2y_2+2y_3+y_4\right)$$

$$=\frac{1}{4*2}\left(1+2\left(\frac{25}{16}\right)+2\left(\frac{36}{16}\right)+2\left(\frac{49}{16}\right)+4\right)$$

$$=\frac{75}{32}=2.34375$$

The exact value $=\int_1^2 x^2\,dx=\frac{1}{3}\left[x^3\right]_1^2=\frac{1}{3}\left[2^3-1^3\right]$

$$=\frac{7}{3}\approx 2.33333$$

The error estimate for Trapezoidal method:

The error from trapezoidal method $=E_T=T-\int_a^b f(x)\,dx$

and computed from $\boxed{|E_T|\leqslant\frac{b-a}{12}h^2 D}$

Where $D=$ upper bound for the value of $|f''(x)|$ on $[a,b]$.

Notice that the error estimate E_T is the maximum predicted error (not actual error).

Ex. Find the upper bound for the error estimate from using the trapezoidal method with $n=10$ for the integral $\int_0^1 x \sin x$.

Sol. $a=0$, $b=1$, $n=10$ $\Longrightarrow h = \frac{1-0}{10} = \frac{1}{10}$

$$f(x) = x \sin x$$

$$f'(x) = x \cos x + \sin x$$

$$f''(x) = x(-\sin x) + \cos x + \cos x$$

$$= 2\cos x - x \sin x$$

$$\Longrightarrow D = 2$$

$$|E_T| \leqslant \frac{b-a}{12} h^2 D$$

$$|E_T| \leqslant \frac{1}{12} \left(\frac{1}{10}\right)^2 (2)$$

$$|E_T| \leqslant \frac{1}{600} \qquad \left[\frac{1}{600} \approx 0.001666\right]$$

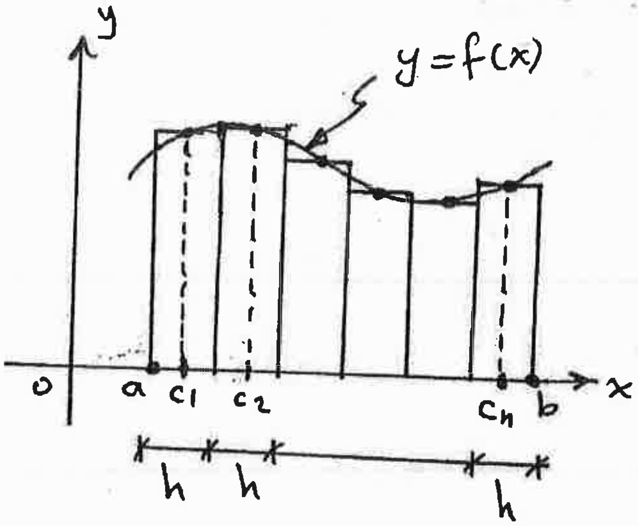
Notice that if $n=100$:

$$|E_T| \leqslant \frac{1}{12} \left(\frac{1}{100}\right)^2 (2)$$

$$|E_T| \leqslant \frac{1}{60000} \qquad \left[\frac{1}{60000} \approx 0.00001666\right]$$

② Midpoint Method:

In this method, the area under the curve is divided into a number of rectangles. The curve intersects each rectangle at the midpoint of the top side.



To estimate $\int_a^b f(x)\,dx$ by midpoint method use:

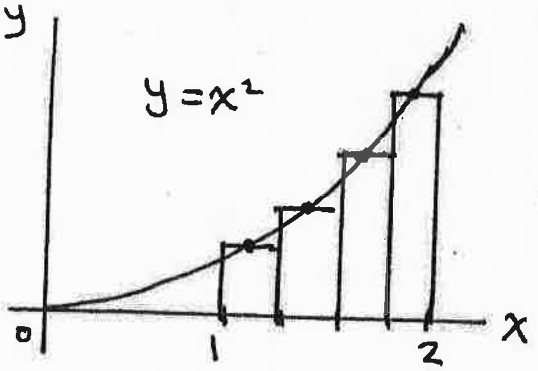
$$\boxed{M = \sum_{k=1}^{n} f(c_k)\cdot h}$$

n : number of rectangles.
c_k : x-coordinate for the midpoint.

Ex. Estimate $\int_1^2 x^2\,dx$ with $n=4$ by midpoint method.

Sol. $a=1$, $b=2$, $n=4$

$$\Rightarrow h = \frac{2-1}{4} = \frac{1}{4}$$



Make a table:

c_k	$f(c_k)$
$c_1 = 1 + \frac{1/4}{2} = \frac{9}{8}$	$f(c_1) = (\frac{9}{8})^2 = \frac{81}{64}$
$c_2 = \frac{9}{8} + \frac{1}{4} = \frac{11}{8}$	$f(c_2) = (\frac{11}{8})^2 = \frac{121}{64}$
$c_3 = \frac{11}{8} + \frac{1}{4} = \frac{13}{8}$	$f(c_3) = \frac{169}{64}$
$c_4 = \frac{13}{8} + \frac{1}{4} = \frac{15}{8}$	$f(c_4) = \frac{225}{64}$

$$\Longrightarrow M = \frac{1}{4}\left[\frac{81}{64} + \frac{121}{64} + \frac{169}{64} + \frac{225}{64}\right] = \frac{149}{64} \doteqdot 2.328125$$

The error estimate for Midpoint method:

$$\boxed{|E_M| \leqslant \frac{b-a}{24} h^2 D}$$

From last example: $\int_1^2 x^2 dx$

$a = 1$, $b = 2$, $h = \frac{1}{4}$

$f(x) = x^2 \Rightarrow f'(x) = 2x \Rightarrow f''(x) = 2 \Rightarrow D = 2$

$$\Rightarrow |E_M| \leqslant \frac{2-1}{24} \left(\frac{1}{4}\right)^2 (2)$$

$$|E_M| \leqslant \frac{1}{192} \qquad \left[\frac{1}{192} = 0.005208 \right]$$

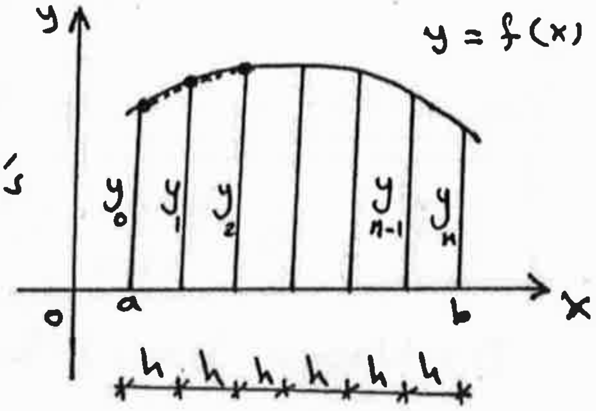
---

③ Simpson's Rule:

Simpson's rule is based on approximating curves with parabolas instead of line segments.

Each three consecutive points are connected with a parabola.

Notice that the number of strips n is even in Simpson's rule.



To approximate $\int_a^b f(x)\,dx$ by Simpson's rule use:

$$\boxed{S = \frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n \right)}$$

We can see that: $y_{odd} * 4$

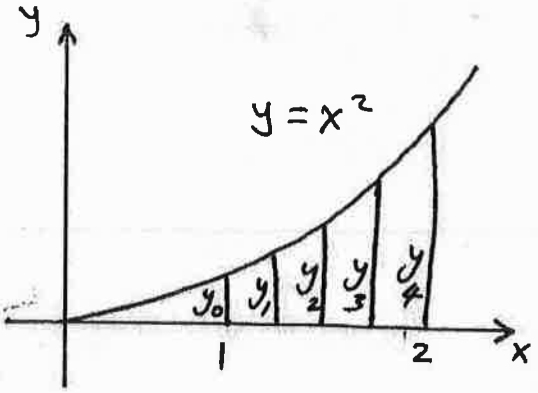
$y_{even} * 2$

y_0 & y_n $* 1$

Ex. Estimate $\int_1^2 x^2\,dx$ with $n=4$ by Simpson's rule.

Sol. $a=1,\ b=2,\ n=4$

$$\Rightarrow h=\frac{2-1}{4}=\frac{1}{4}$$



Make a table:

x	y
$x_0=1$	$y_0=1$
$x_1=\frac{5}{4}$	$y_1=25/16$
$x_2=\frac{6}{4}$	$y_2=36/16$
$x_3=\frac{7}{4}$	$y_3=49/16$
$x_4=2$	$y_4=4$

this table is like the table of the example of trapezoidal method (P.16)

$$S=\frac{h}{3}\left[y_0+4y_1+2y_2+4y_3+y_4\right]$$

$$=\frac{1}{4*3}\left[1+4\left(\frac{25}{16}\right)+2\left(\frac{36}{16}\right)+4\left(\frac{49}{16}\right)+4\right]$$

$$=\frac{7}{3}\approx 2.333333$$ [exact value]

The error estimate for Simpson's rule:

$$\boxed{|E_S|\leqslant\frac{b-a}{180}h^4D}$$

D: Upper bound for the value of $|f^{(4)}|$ on $[a,b]$.

Notice that when the power of $f(x)$ is $\leqslant 3$, the value of D becomes zero and the error becomes zero.

Therefore S gives exact value for the integrals of the functions of third degree or less.

For the last example: $f(x)=x^2 \Rightarrow f'(x)=2x$

$\Rightarrow f''(x)=2 \Rightarrow f'''(x)=0 \Rightarrow f^{(4)}(x)=0 \Rightarrow D=0$

$\Rightarrow$ error $=0$

Determining the number of steps n that guarantee a given accuracy:

Ex. Determine n that will guarantee an accuracy of at least 10^{-7} for using:

① Trapezoidal rule.

② Simpson's rule.

To approximate $\int_2^4 x^4\,dx$.

sol.

① By trapezoidal rule:

$$|E_T| \leqslant \frac{b-a}{12} h^2 D$$

$a=2$, $b=4$, $h=\frac{b-a}{n}=\frac{2}{n}$

$f(x)=x^4 \Rightarrow f'(x)=4x^3 \Rightarrow f''(x)=12x^2$

at $x=4 \Rightarrow f(4)=12(4)^2=192 \Rightarrow D=192$

$$\Rightarrow |E_T| \leqslant \frac{4-2}{12}\left(\frac{2}{n}\right)^2 * 192$$
$$\leqslant \frac{128}{n^2}$$

$\Rightarrow \frac{128}{n^2} \leqslant 10^{-7} \Rightarrow n^2 \geqslant 128 * 10^7$

$\Rightarrow n \geqslant 35777.08 \Rightarrow n=35778$

② By Simpson's rule:

$$|E_s| \leqslant \frac{b-a}{180} h^4 D$$

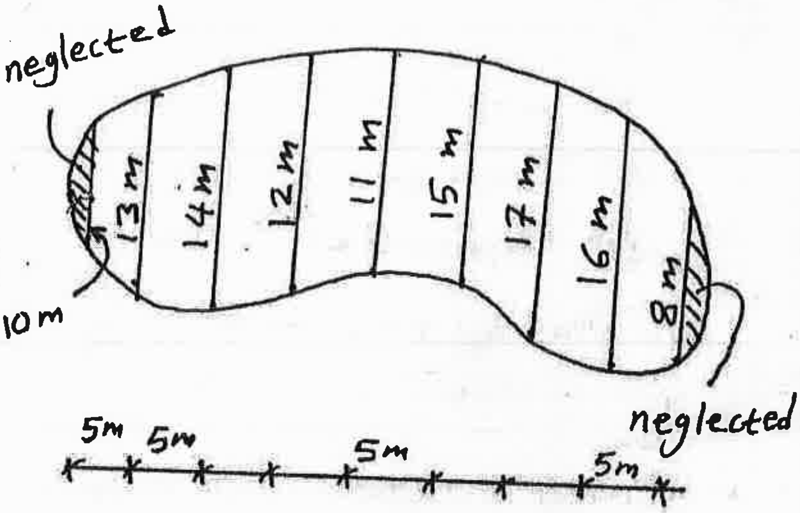
$f''' = 24x \Rightarrow f^{(4)}=24 \Rightarrow D=24$

$\Rightarrow |E_s| \leqslant \frac{2}{180}\left(\frac{2}{n}\right)^4 * 24 \Longrightarrow \leqslant \frac{64}{15n^4}$

$\frac{64}{15n^4} \leqslant 10^{-7} \Rightarrow n \geqslant 80.82 \Rightarrow n=82$ (even).

Ex. A town wants to drain a swamp and then fill it with dirt. The average depth of swamp is 3 m. How many cubic meters of dirt required to fill the swamp?

Sol.



Volume of swamp

= area of top surface * average depth

Use Simpson's rule:

$h = 5m$

$$S = \frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + 2y_6 + 4y_7 + y_8\right]$$

$$= \frac{5}{3}\left[10 + 4(13) + 2(14) + 4(12) + 2(11) + 4(15) + 2(17) + 4(16) + 8\right]$$

$$= 543.33 \ m^2$$

$\Rightarrow$ Required filling dirt $= 543.33 * 3 = 1629.99 \ m^3$

Ex. The table below shows the velocity of submarine with the traveling time. Use simpson's rule to estimate the distance traveled during the 10 hours peroid.

Sol. $v = \frac{ds}{dt} \Rightarrow ds = v \cdot dt$

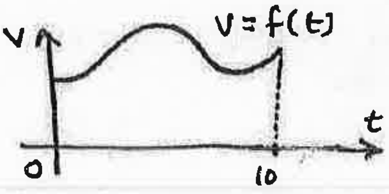
$$\Longrightarrow S = \int_0^{10} v(t) \cdot dt$$

t (hr)	V (mph)
0	12
1	14
2	17
3	21
4	22
5	21
6	15
7	11
8	11
9	14
10	17

Use Simpson's rule : $h = 1$, $n = 10$

$$\Rightarrow S = \frac{1}{3}\left[12 + 4(14) + 2(17) + 4(21) + 2(22) + 4(21) + 2(15) + 4(11) + 2(11) + 4(14) + 17\right]$$

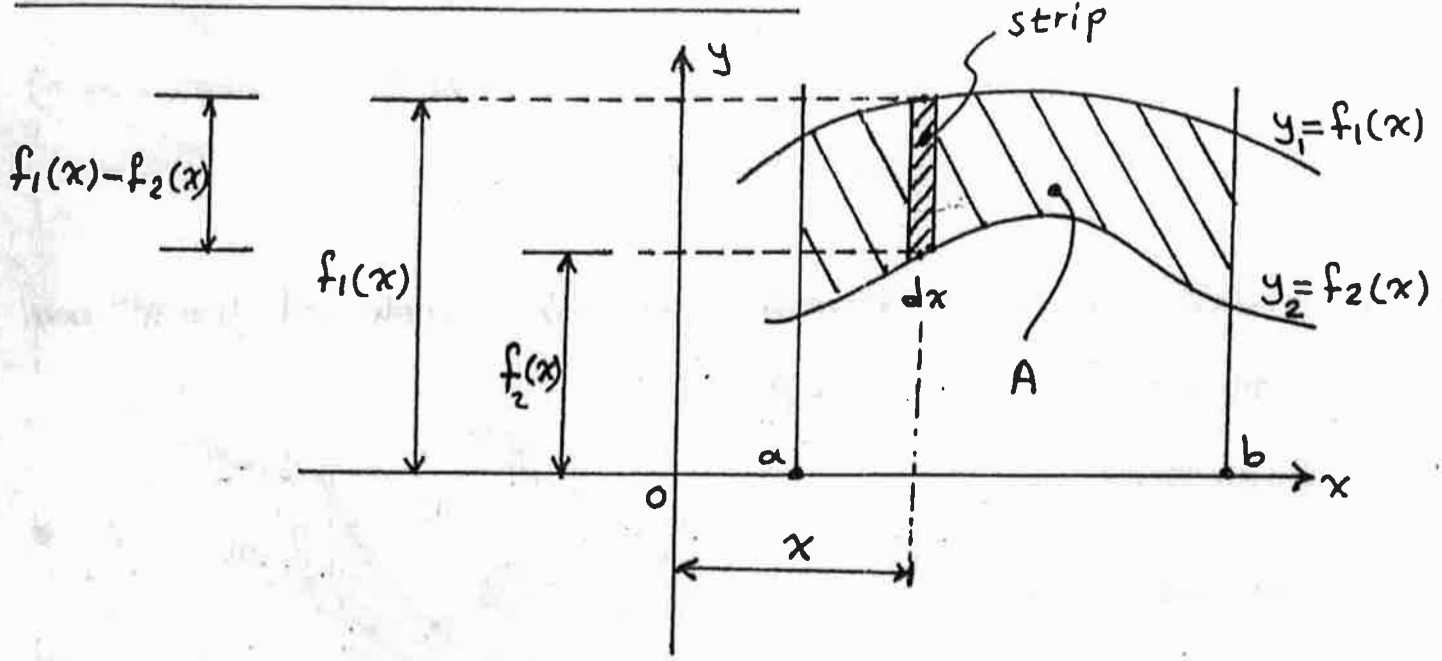
$= 161$ mile



CHAPTER SIX

APPLICATIONS OF DEFINITE INTEGRALS

6.1 : Areas between Curves :



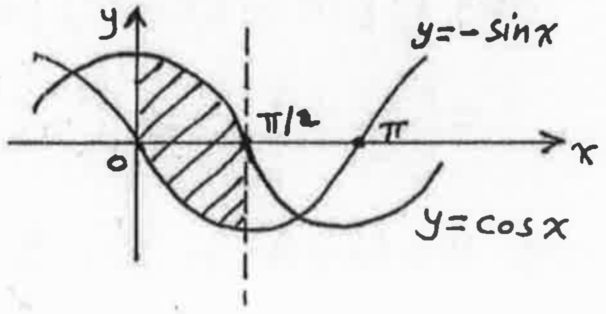
We want to find the area between the graphs of the continuous functions $y_1 = f_1(x)$ and $y_2 = f_2(x)$.

Area of strip = Area of rectangle = $[f_1(x) - f_2(x)] * dx$

If the area between the upper curve $y_1 = f_1(x)$ and the lower curve $y_2 = f_2(x)$ is denoted by A, then :

$$\boxed{A = \int_a^b [f_1(x) - f_2(x)]\, dx}$$

Ex.1 : Find the area between the curves $y = \cos x$ and $y = -\sin x$ from $x = 0$ to $x = \pi/2$.



Sol. From the graph, we can see that :

$f_1(x) = \cos x$

$f_2(x) = -\sin x$

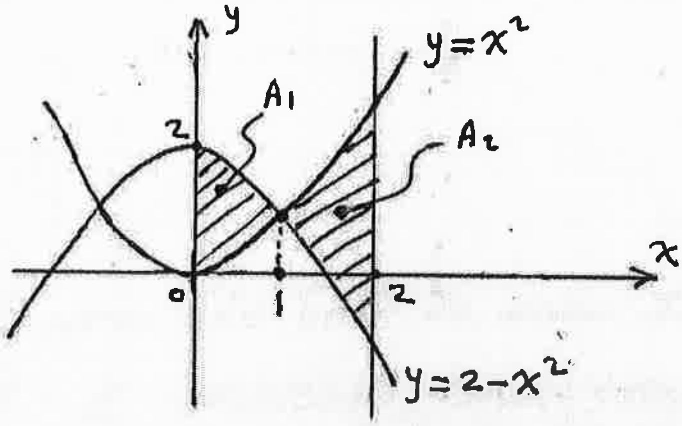
$$\text{Area} = \int_a^b [f_1(x) - f_2(x)]\,dx$$

$$\Rightarrow \text{Area} = \int_0^{\pi/2} [\cos x - (-\sin x)]\,dx = \int_0^{\pi/2} (\cos x + \sin x)\,dx$$

$$= [\sin x - \cos x]_0^{\pi/2} = (\sin\tfrac{\pi}{2} - \cos\tfrac{\pi}{2}) - (\sin 0 - \cos 0)$$

$$= (1-0) - (0-1) = 2 \text{ units}$$

Ex.2 : Find the area bounded by the graphs of $y = x^2$ and $y = 2 - x^2$ for $0 \leqslant x \leqslant 2$.



Sol. First find the point of intersection between the two graphs.

$y_1 = y_2$

$x^2 = 2 - x^2 \Rightarrow 2x^2 = 2$

$\Rightarrow x^2 = 1 \Rightarrow x = \mp 1$

$x = -1$ out of interval

$\Rightarrow x = 1$ is the x-coordinate of the point of intersection.

From $x = 0$ to $x = 1$: $f_1(x) = 2 - x^2$ and $f_2(x) = x^2$.

From $x = 1$ to $x = 2$: $f_1(x) = x^2$ and $f_2(x) = 2 - x^2$.

$$A_1 = \int_0^1 [(2-x^2) - x^2]\,dx = \int_0^1 (2 - 2x^2)\,dx$$

$$= [2x - \tfrac{2}{3}x^3]_0^1 = [2(1) - \tfrac{2}{3}(1)^3] - [2(0) - \tfrac{2}{3}(0)^3] = \frac{4}{3} \text{ units}$$

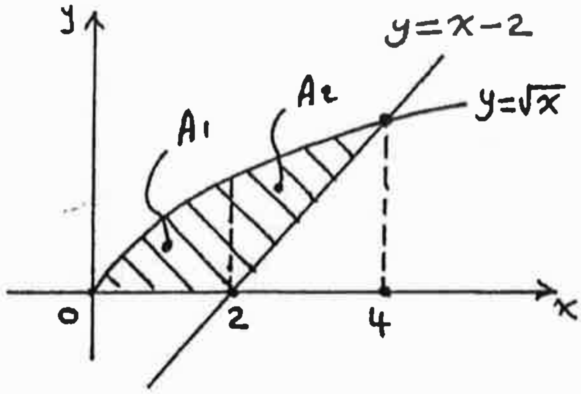
$$A_2 = \int_1^2 [x^2 - (2-x^2)]\,dx = \int_1^2 (2x^2 - 2)\,dx$$

$$= [\tfrac{2}{3}x^3 - 2x]_1^2 = [\tfrac{2}{3}(2)^3 - 2(2)] - [\tfrac{2}{3}(1)^3 - 2(1)] = \frac{8}{3} \text{ units}$$

$$\Rightarrow A = A_1 + A_2 = \frac{4}{3} + \frac{8}{3} = \frac{12}{3} = 4 \text{ units.}$$

Ex.3: Find the area of the region in the first quadrant bounded above by the curve $y=\sqrt{x}$ and below by the x-axis and the line $y=x-2$.

Sol. the point of intersection:



$$y_1 = y_2$$

$$\sqrt{x} = x-2$$

squaring $\Rightarrow$ $x = (x-2)^2$

$$x = x^2 - 4x + 4$$

$$x^2 - 5x + 4 = 0$$

$$(x-4)(x-1) = 0$$

$$\Rightarrow x = 4 \quad \text{and} \quad x = 1$$

$x=1$ is extraneous root comes from squaring (neglected).

$\Rightarrow$ $x=4$ is the x-coordinate for the intersection point.

From $x=0$ to $x=2$ the area is under the curve $y=\sqrt{x}$.

From $x=2$ to $x=4$: $f_1(x) = \sqrt{x}$ and $f_2(x) = x-2$.

$$A_1 = \int_0^2 \sqrt{x}\,dx = \frac{2}{3}\left[x^{3/2}\right]_0^2 = \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right]$$

$$= \frac{2}{3} * \sqrt{8} = \frac{2}{3} * 2\sqrt{2} = \frac{4\sqrt{2}}{3} \quad \text{units}$$

$$A_2 = \int_2^4 [\sqrt{x} - (x-2)]\,dx = \int_2^4 (x^{1/2} - x + 2)\,dx$$

$$= \left[\frac{2}{3}x^{3/2} - \frac{1}{2}x^2 + 2x\right]_2^4 = \left[\frac{2}{3}4^{3/2} - \frac{1}{2}4^2 + 2(4)\right] - \left[\frac{2}{3}2^{3/2} - \frac{2^2}{2} + 2(2)\right]$$

$$= \left(\frac{16}{3}\right) - \left(\frac{4\sqrt{2}}{3} + 2\right) = \frac{10}{3} - \frac{4\sqrt{2}}{3}$$

$$\Rightarrow A = A_1 + A_2 = \frac{4\sqrt{2}}{3} + \left(\frac{10}{3} - \frac{4\sqrt{2}}{3}\right) = \frac{10}{3} \quad \text{units}$$

Note: you can find the area by another way:

$A = $ (Area under $y=\sqrt{x}$ from $x=0$ to $x=4$) $-$ triangle area

$$= \int_0^4 \sqrt{x}\,dx - \frac{1}{2}(2)(2) = \frac{10}{3} \text{ units}$$

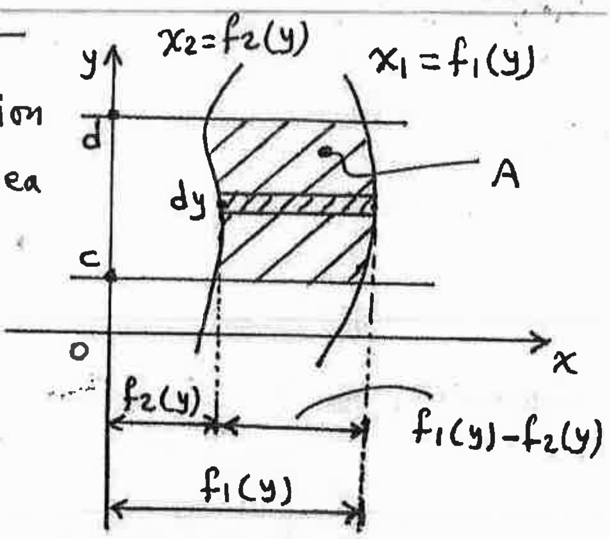
Integrating with respect to y :

When the bounding curves for a region are described by $x=f(y)$, the area becomes :

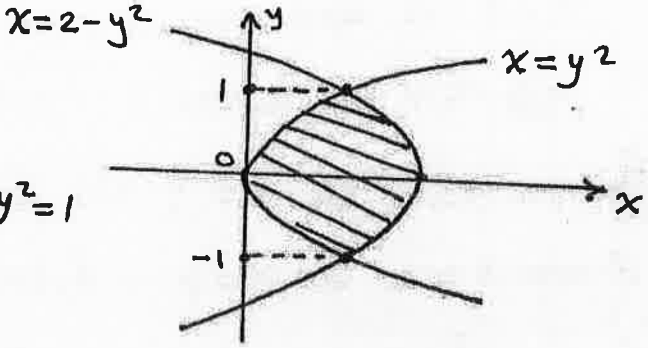
$$\boxed{A = \int_c^d [f_1(y)-f_2(y)]\,dy}$$

where : $f_1(y)$ is right hand curve .

and $f_2(y)$ is left hand curve .



Ex.1 : Find the area bounded by the graphs of $x=y^2$ and $x=2-y^2$.



sol. points of intersection:

$x_1=x_2$

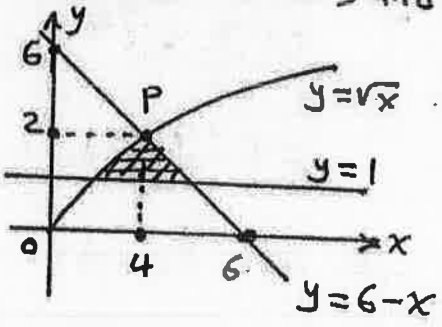
$y^2=2-y^2 \Rightarrow 2y^2=2 \Rightarrow y^2=1$

$\Rightarrow y=-1$ & $y=1$

$$\Rightarrow \text{Area} = \int_{-1}^{1}[(2-y^2)-y^2]\,dy$$

$$= \int_{-1}^{1}(2-2y^2)\,dy = \left[2y-\frac{2}{3}y^3\right]_{-1}^{1} = (2-\tfrac{2}{3})-(-2+\tfrac{2}{3}) = \frac{8}{3} \text{ units}$$

Ex.2 : Find the area of the region bounded on the left by the curve $y=\sqrt{x}$, on the right by the line $y=6-x$, and below by the line $y=1$.



sol. $y_1=y_2 \Rightarrow \sqrt{x}=6-x$

$\Rightarrow x=(6-x)^2 \Rightarrow x=36-12x+x^2$

$\Rightarrow x^2-13x+36=0 \Rightarrow (x-4)(x-9)=0$

$\Rightarrow x=4$ & $x=9$ (neglected)

at $x=4 \Rightarrow y=\sqrt{4}=2$ (or $y=6-4=2$)

$y=\sqrt{x} \Rightarrow y^2=x \Rightarrow x=y^2$ is $f_2(y)$

$y=6-x \Rightarrow x=6-y$ is $f_1(y)$

$\Rightarrow A=\int_1^2[(6-y)-y^2]\,dy = 13/6$ units

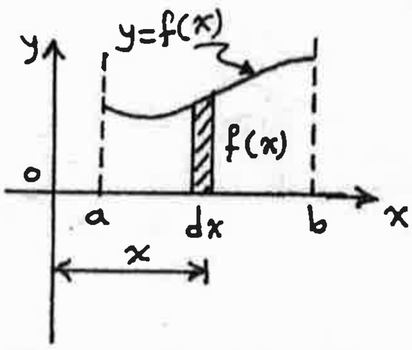
6.2 : Volumes of Solids of Revolution :

A solid of revolution is a solid whose shape can be generated by revolving plane region about an axis.

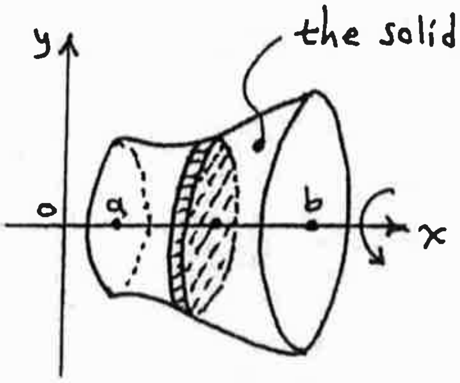
We will study three methods to calculate the volumes of these solids : Disk , Washer , and Cylindrical shell methods.

① The Disk Method:

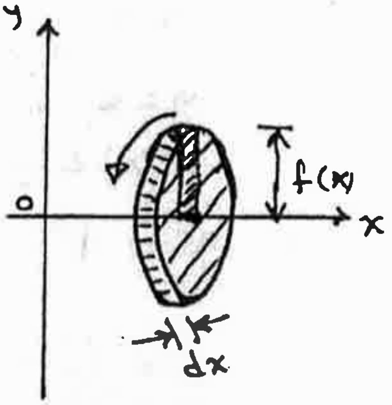
This method is used when the region is border on the axis of revolution.



before revolving



after revolving about x-axis



the disk

Disk volume = Area of base * height

= Area of circle * thickness

$= \pi \, (f(x))^2 * dx$

If the radius of the circle = $r(x)$, then :

Disk volume = $\pi \, (r(x))^2 * dx$

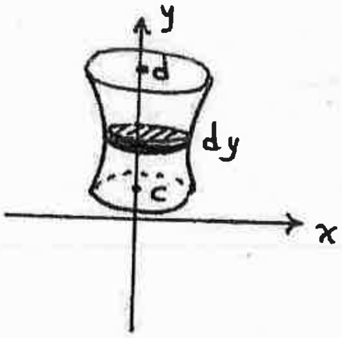
If the volume of the solid = V

$$\Longrightarrow \boxed{V = \int_a^b \pi \, (r(x))^2 \, dx}$$

the formula of the volume when the region is revolved about a horizontal axis.

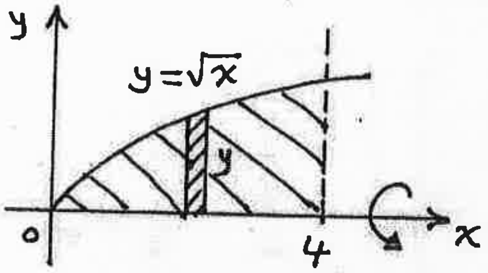
If the region is revolved about a vertical axis, the volume becomes:

$$\boxed{V = \int_c^d \pi\,(r(y))^2\,dy}$$



Ex.1: The region between the curve $y=\sqrt{x}$, $0 \leqslant x \leqslant 4$, and the x-axis is revolved about the x-axis to generate a solid. Find the volume of this solid.

Sol. $V = \int_a^b \pi\,(r(x))^2 . dx$



$a=0$, $b=4$

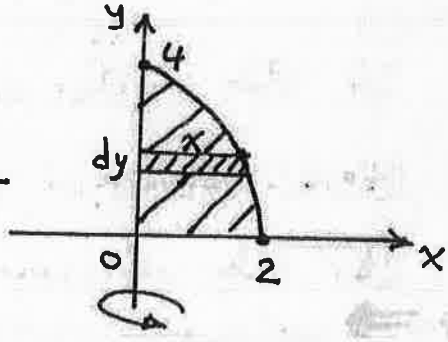
$r(x) = y = \sqrt{x}$

$\Rightarrow V = \int_0^4 \pi\,(\sqrt{x})^2 . dx$

$= \int_0^4 \pi x\,dx = \frac{\pi}{2}\left[x^2\right]_0^4 = \frac{\pi}{2}\left[4^2 - 0\right] = 8\pi$ cubic units

Ex.2: Find the volume of the solid generated by revolving the curve $y = 4 - x^2$, $0 \leqslant x \leqslant 2$, about y-axis.

Sol. $V = \int_c^d (r(y))^2 . dy$



$c=0$, $d=4$

$y = 4 - x^2 \Rightarrow x^2 = 4 - y \Rightarrow x = \mp\sqrt{4-y}$

$\Rightarrow x = \sqrt{4-y}$ the right portion

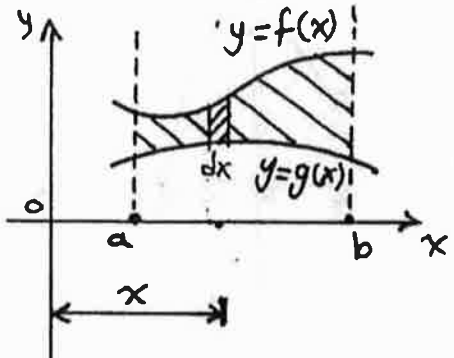
$\Rightarrow r(y) = \sqrt{4-y}$

$\Rightarrow V = \int_0^4 \pi\,(\sqrt{4-y})^2 dy = \int_0^4 \pi\,(4-y)\,dy$

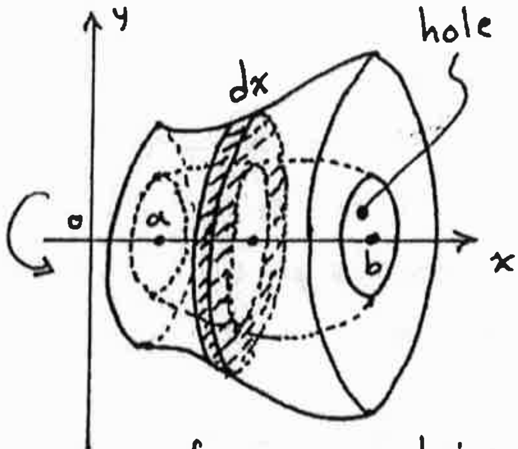
$= \pi\left[4y - \frac{y^2}{2}\right]_0^4 = \pi\left[(16 - \frac{16}{2}) - (0)\right] = 8\pi$ cubic units

② The Washer Method:

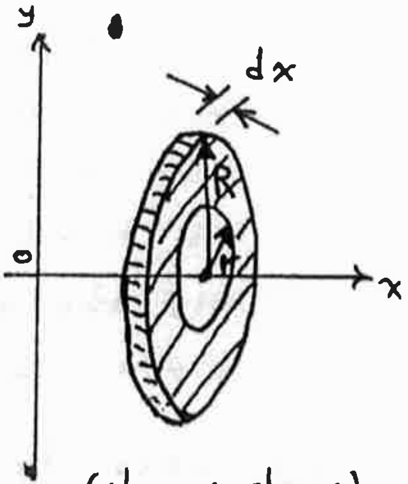
If the region is not border on the axis of revolution, the solid will has a hole (or cavity). In this case, the washer method is used.



before revolving



after revolving about x-axis



(the washer)
R: outer radius
r: inner radius

$$\text{Washer volume} = \text{Area of base} * \text{height}$$
$$= (\pi R^2 - \pi r^2) * dx$$
$$= \pi (R^2 - r^2) * dx$$
$$= \pi [(f(x))^2 - (g(x))^2] * dx$$

We can see that $R = R(x)$ and $r = r(x)$.

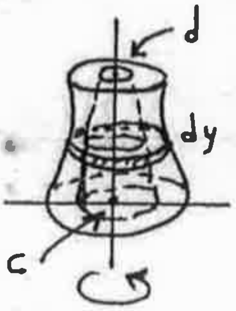
$\Rightarrow$ Washer volume $= \pi \; [(R(x))^2 - (r(x))^2] * dx$

If the volume of the solid $= V$

$$\Rightarrow \boxed{V = \int_a^b \pi \; [(R(x))^2 - (r(x))^2] \, dx}$$

If the region is revolved about a vertical axis:

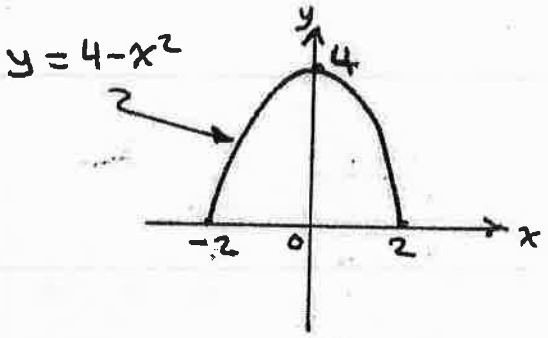
$$\Rightarrow \boxed{V = \int_c^d \pi \; [(R(y))^2 - (r(y))^2] \, dy}$$



Ex. For the region bounded by the graph $y=4-x^2$ and the x-axis, find the volumes of the solids obtained by revolving the region about:

① The line $y=-3$ ② The line $x=3$.

Sol. First find the x-intercepts:



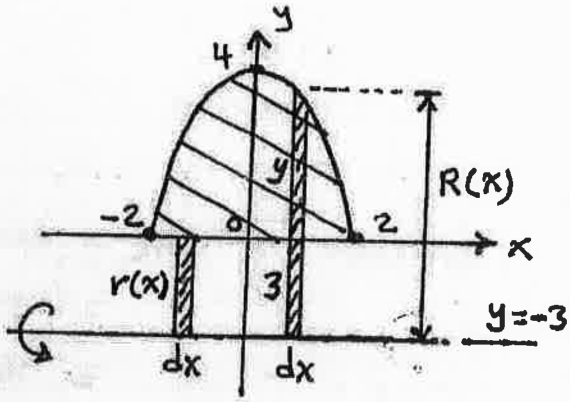
$y=0 \Rightarrow 4-x^2=0 \Rightarrow x^2=4$

$\Rightarrow x=2$ & $x=-2$

and the y-intercept:

$x=0 \Rightarrow y=4$

① About the line $y=-3$:



$$V = \int_a^b \pi\left[(R(x))^2-(r(x))^2\right]dx$$

$a=-2$, $b=2$

$R(x)= y+3 = (4-x^2)+3 = 7-x^2$

$r(x)= 3$

$$\Rightarrow V = \int_{-2}^{2} \pi\left[(7-x^2)^2-(3)^2\right]dx$$

$$= \pi \int_{-2}^{2} \left[(49-14x^2+x^4)-9\right]dx$$

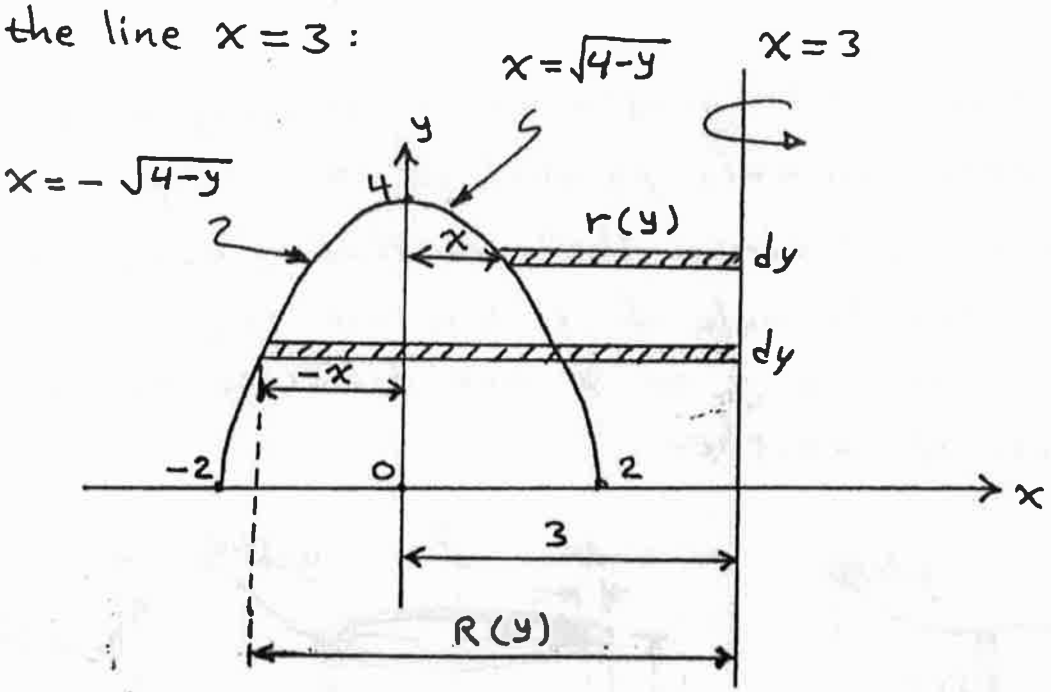
$$= \pi \int_{-2}^{2} (x^4-14x^2+40)\,dx$$

$$= \pi \left[\frac{x^5}{5}-\frac{14x^3}{3}+40x\right]_{-2}^{2}$$

$$= \pi\left[\left(\frac{32}{5}-\frac{112}{3}+80\right)-\left(\frac{-32}{5}+\frac{112}{3}-80\right)\right]$$

$$= \frac{1472}{15}\pi \text{ cubic units.}$$

② About the line $x = 3$:



$$V = \int_c^d \pi\left[(R(y))^2 - (r(y))^2\right] dy$$

$c = 0 \;,\; d = 4$

$y = 4 - x^2 \Rightarrow x^2 = 4 - y \Rightarrow x = \mp\sqrt{4-y}$

$x = \sqrt{4-y}$ right half.

$x = -\sqrt{4-y}$ left half.

$R(y) = 3 + |-x| = 3 + x = 3 + \sqrt{4-y}$

$r(y) = 3 - x = 3 - \sqrt{4-y}$

$$\Rightarrow V = \pi\int_0^4\left[(3+\sqrt{4-y})^2 - (3-\sqrt{4-y})^2\right] dy$$

$$= \pi\int_0^4\left\{\left[9 + 6\sqrt{4-y} + (4-y)\right] - \left[9 - 6\sqrt{4-y} + (4-y)\right]\right\} dy$$

$$= \pi\int_0^4\left[12(4-y)^{1/2}\right] dy = -12\pi\int_0^4 (4-y)^{1/2} \cdot -dy$$

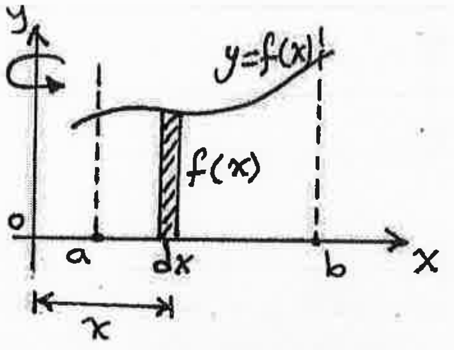
$$= -12\pi\left[\frac{(4-y)^{3/2}}{3/2}\right]_0^4 = -8\pi\left[0 - 4^{3/2}\right]$$

$$= 64\pi \text{ cubic units.}$$

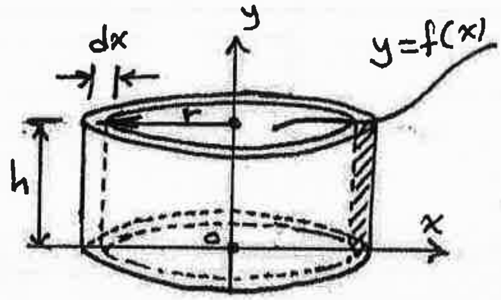
6.3: Cylindrical Shells:

Cylindrical shell is generated by revolving a rectangular strip about an axis parallel to this strip.

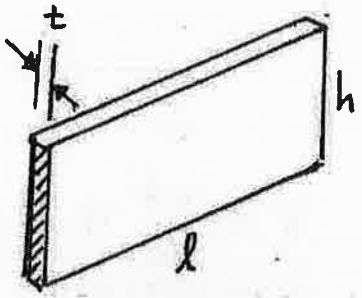
Sometimes, cylindrical shells method is easier to use because the formula of it does not requires squaring, or in cases where y or x are difficult to determined in terms of another.



before revolving



after revolving (the cylindrical shell)



Flatting the cylindrical shell (rectangular sheet)

Volume of cylindrical shell = Volume of rectangular sheet

$$= l * h * t$$

$$= 2\pi r * h * dx$$

We can see that $r = r(x)$ and $h = h(x)$

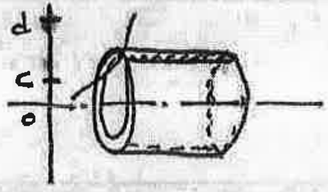
$\Rightarrow$ volume of cylindrical shell $= 2\pi * r(x) * h(x) * dx$

If the volume of the solid $= V$

$$\Rightarrow \boxed{V = \int_a^b 2\pi \cdot r(x) \cdot h(x) \cdot dx}$$ revolving about vertical axis

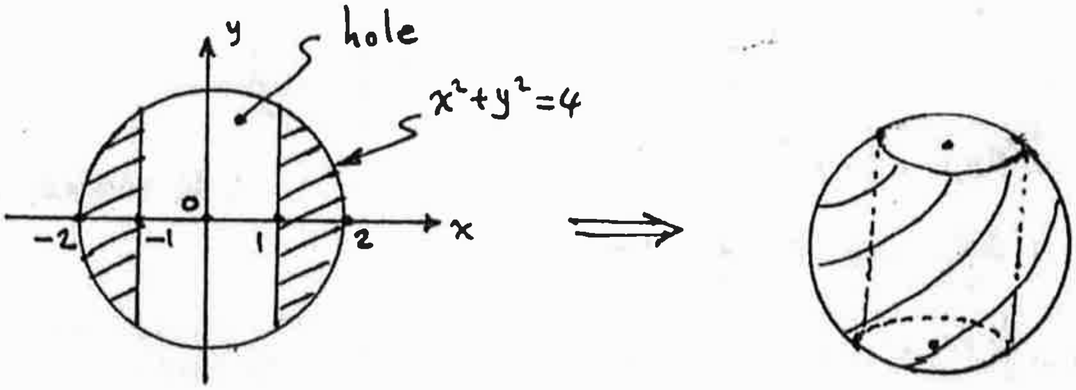
When the region is revolved about horizontal axis:

$$\Rightarrow \boxed{V = \int_c^d 2\pi \cdot r(y) \cdot h(y) \cdot dy}$$

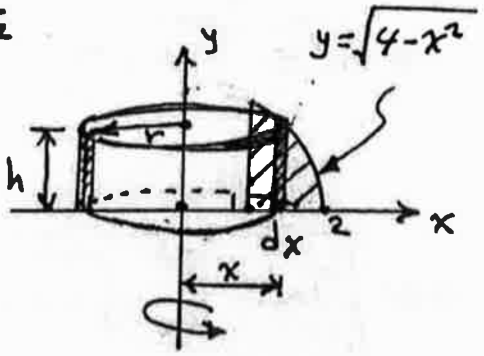


Ex. The disk enclosed by the circle $x^2+y^2=4$ is revolved about the y-axis to generate a solid sphere. A hole of diameter 2 units is then bored through the sphere along the y-axis. Find the volume of the cored sphere.

sol.



Take only the portion of the circle in the 1st quadrant.



$x^2+y^2=4 \Rightarrow y^2=4-x^2 \Rightarrow y=\mp\sqrt{4-x^2}$

$\Rightarrow y=\sqrt{4-x^2}$ is the upper portion.

$V=\int_a^b 2\pi \cdot r(x) \cdot h(x) \cdot dx$

$a=1 \ , \ b=2$

$r(x)=x$

$h(x)=f(x)=\sqrt{4-x^2}$

$$\Rightarrow V=\int_1^2 2\pi \cdot x \cdot \sqrt{4-x^2} \cdot dx$$

$$=-\pi\int_1^2 (4-x^2)^{1/2} \cdot -2x\,dx = -\pi\left[\frac{(4-x^2)^{3/2}}{3/2}\right]_1^2$$

$$=\frac{-2}{3}\pi\left[(4-x^2)^{3/2}\right]_1^2$$

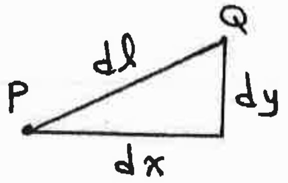
$$=\frac{-2}{3}\pi\left[(4-2^2)^{3/2}-(4-1^2)^{3/2}\right]=\frac{-2}{3}\pi\left[-\sqrt{27}\right]$$

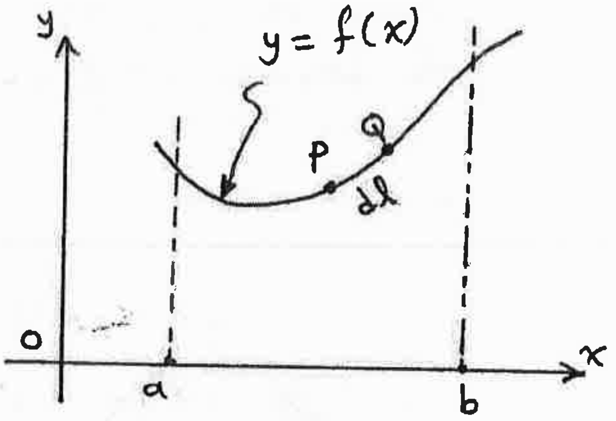
$$=\frac{-2}{\cancel{3}}\pi\left[-\cancel{3}\sqrt{3}\right] \doteq 2\sqrt{3}\,\pi$$ cubic units (volume of upper portion

$\Rightarrow$ Volume of cored sphere $=2 * 2\sqrt{3}\,\pi = 4\sqrt{3}\,\pi$ cubic units

6.4: Lengths of Curves in the Plane:

Take a small portion on the graph (the segment PQ).





(Q closed to P)

$$dl = \sqrt{(dy)^2 + (dx)^2}$$

but $f'(x) = \frac{dy}{dx} \Rightarrow dy = f'(x) \, . \, dx$

$$\Rightarrow dl = \sqrt{(f'(x).dx)^2 + (dx)^2} = \sqrt{(dx)^2 \, [(f'(x))^2 + 1]}$$

$$\Rightarrow dl = \sqrt{(f'(x))^2 + 1} \; dx = \sqrt{1 + (f'(x))^2} \; dx$$

$$\Rightarrow dl = \sqrt{1 + (\frac{dy}{dx})^2} \; dx$$

If the length of the curve from $x = a$ to $x = b$ is L:

$$\Rightarrow \boxed{L = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2} \; dx}$$

If $\frac{dy}{dx}$ fails to exist and $\frac{dx}{dy}$ may be exist, then:

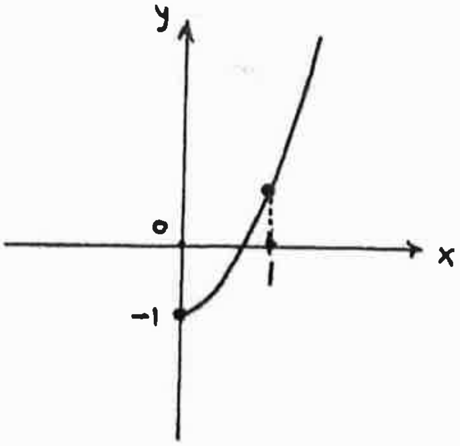
$$\boxed{L = \int_c^d \sqrt{1 + (\frac{dx}{dy})^2} \; dy}$$

The short differential formula for the length of the curve is:

$$\boxed{L = \int dl} \quad , \quad dl = \sqrt{(dx)^2 + (dy)^2}$$

Ex.1: Find the length of the curve $y=\frac{4\sqrt{2}}{3}x^{3/2}-1$, $0\leqslant x\leqslant 1$

Sol. $L=\int_a^b\sqrt{1+(\frac{dy}{dx})^2}\,.\,dx$



$a=0$, $b=1$

$$\frac{dy}{dx}=\frac{4\sqrt{2}}{3}*\frac{3}{2}*x^{1/2}=2\sqrt{2}\,x^{1/2}$$

$$(\frac{dy}{dx})^2=4*2*x=8x$$

$$\sqrt{1+(\frac{dy}{dx})^2}=(1+8x)^{1/2}$$

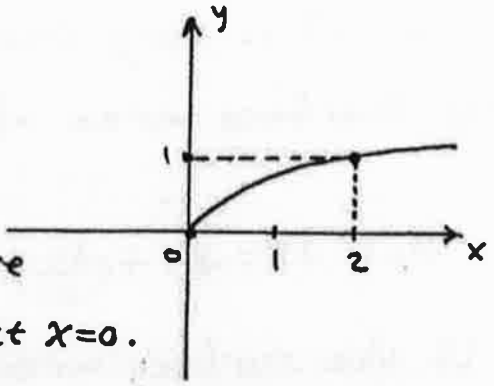
$$\Rightarrow L=\int_0^1(1+8x)^{1/2}\,.\,dx=\frac{1}{8}\int_0^1(1+8x)^{1/2}.8dx$$

$$=\frac{1}{8}\cdot\frac{2}{3}\left[(1+8x)^{3/2}\right]_0^1=\frac{1}{12}\left[(1+8)^{3/2}-(1+0)^{3/2}\right]$$

$$=\frac{1}{12}\left[9^{3/2}-1^{3/2}\right]=\frac{1}{12}\left[27-1\right]=\frac{26}{12}=\frac{13}{6}\text{ units.}$$

Ex.2: Find the length of the curve $y=(\frac{x}{2})^{2/3}$, $0\leqslant x\leqslant 2$.

Sol. $\frac{dy}{dx}=\frac{2}{3}(\frac{x}{2})^{-1/3}(\frac{1}{2})=\frac{1}{3}(\frac{x}{2})^{-1/3}$



$$(\frac{dy}{dx})^2=\frac{1}{9}(\frac{x}{2})^{-2/3}=\frac{1}{9}(\frac{2}{x})^{2/3}$$

$(\frac{dy}{dx})^2$ is not defined at $x=0$, therefore the integrand $\sqrt{1+(\frac{dy}{dx})^2}$ is not defined at $x=0$.

So try $\frac{dx}{dy}$:

$$y=(\frac{x}{2})^{2/3}\Rightarrow y^{3/2}=\frac{x}{2}\Rightarrow x=2y^{3/2}\Rightarrow\frac{dx}{dy}=2(\frac{3}{2})y^{1/2}=3y^{1/2}$$

$$\Rightarrow\sqrt{1+(\frac{dx}{dy})^2}=\sqrt{1+(3y^{1/2})^2}=(1+9y)^{1/2}$$

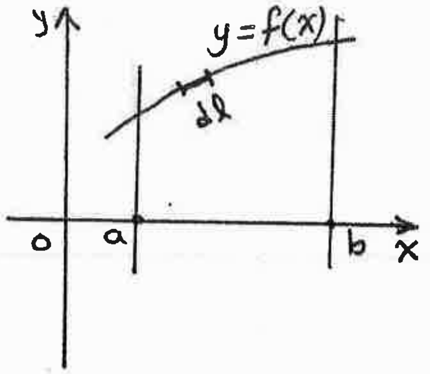
at $x=0\Rightarrow y=(\frac{0}{2})^{2/3}=0\Rightarrow c=0$

at $x=2\Rightarrow y=(\frac{2}{2})^{2/3}=1\Rightarrow d=1$

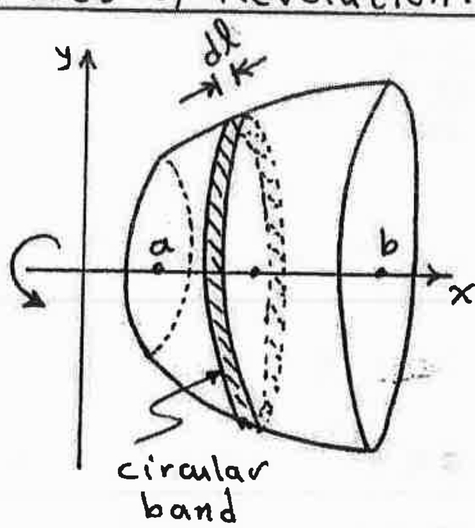
$$\Rightarrow L=\int_c^d\sqrt{1+(\frac{dx}{dy})^2}\,dy=\int_0^1(1+9y)^{1/2}dy=\frac{1}{9}\int_0^1(1+9y)^{1/2}.9dy$$

$$=\frac{1}{9}*\frac{2}{3}\left[(1+9y)^{3/2}\right]_0^1=\frac{2}{27}(10\sqrt{10}-1)\approx 2.268\text{ units}$$

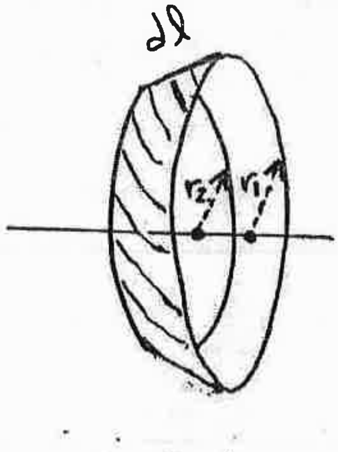
6.5 : Areas of Surfaces of Revolution:



(before revolving)



(after revolving)

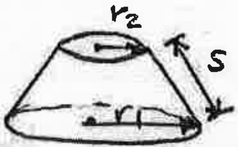


(the circular band)

Surface area of the circular band = Surface area of frustum

Surface area of frustum $= 2\pi \left(\frac{r_1+r_2}{2}\right) \cdot s$

$= \pi (r_1+r_2) \cdot s$



$\Rightarrow$ Surface area of the circular band $= \pi (r_1+r_2) \cdot d\ell$

As $d\ell$ is very small, then $r_1 = r_2 = r$

$\Rightarrow$ Surface area of the circular band $= \pi (r+r) \cdot d\ell$

$= 2\pi r \cdot d\ell$

But: $d\ell = \sqrt{1+(\frac{dy}{dx})^2}\, dx = \sqrt{1+(\frac{dx}{dy})^2}\, dy$, and $r = r(x) = r(y)$.

If the surface area of the revolved curve $= S$, then :

$$\boxed{S = \int_a^b 2\pi \cdot r(x) \cdot \sqrt{1+(\frac{dy}{dx})^2}\, dx}$$ ——— ① For revolving about horizontal or vertical axis

$$\boxed{S = \int_c^d 2\pi \cdot r(y) \cdot \sqrt{1+(\frac{dx}{dy})^2}\, dy}$$ ——— ② For revolving about horizontal or vertical axis

Notes:

* If the curve is described in terms of x use eq. ①.

* If the curve is described in terms of y use eq. ②.

The short differential formula for the surface area is:

$$\boxed{S = \int 2\pi \rho \, dl}$$ revolving about horizontal or vertical axis

If $\rho = r(x) \Rightarrow$ the limits of integration are a & b.

If $\rho = r(y) \Rightarrow$ the limits of integration are c & d.

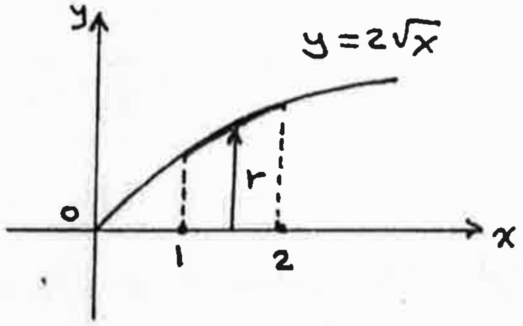
$dl = \sqrt{1 + (\frac{dy}{dx})^2}\, dx$ for using $r(x)$.

$dl = \sqrt{1 + (\frac{dx}{dy})^2}\, dy$ for using $r(y)$.

Ex.1: Find the surface area generated by revolving the curve $y = 2\sqrt{x}$, $1 \leq x \leq 2$, about the x-axis.

Sol.

$$S = \int_a^b 2\pi \cdot r(x) \cdot \sqrt{1 + (\frac{dy}{dx})^2}\, dx$$



$a = 1$, $b = 2$

$r(x) = y = 2\sqrt{x}$

$$\frac{dy}{dx} = 2\left(\frac{1}{2\sqrt{x}}\right) = \frac{1}{\sqrt{x}}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(\frac{1}{\sqrt{x}}\right)^2 = \frac{1}{x} \Rightarrow \sqrt{1 + (\frac{dy}{dx})^2} = \sqrt{1 + \frac{1}{x}}$$

$$\Rightarrow S = \int_1^2 2\pi \cdot 2\sqrt{x} \cdot \sqrt{1 + \frac{1}{x}}\, dx$$

$$= 4\pi \int_1^2 \sqrt{x} \cdot \sqrt{\frac{x+1}{x}}\, dx = 4\pi \int_1^2 \cancel{\sqrt{x}} \cdot \frac{\sqrt{x+1}}{\cancel{\sqrt{x}}} \cdot dx$$

$$= 4\pi \int_1^2 (x+1)^{1/2} \cdot dx = 4\pi \cdot \frac{2}{3} \left[(x+1)^{3/2}\right]_1^2$$

$$= \frac{8}{3}\pi \left(3\sqrt{3} - 2\sqrt{2}\right) \text{ units}$$

Ex.2: Find the surface area generated by revolving the curve $x=\frac{1}{3}(y^2+2)^{3/2}$, $1\leqslant y\leqslant 2$, about the x-axis

Sol. $S=\int_c^d 2\pi r(y)\sqrt{1+(\frac{dx}{dy})^2}\,dy$

$c=1$ & $d=2$

$r(y)=y$

$\frac{dx}{dy}=\frac{1}{\cancel{3}}(\frac{\cancel{3}}{\cancel{2}})(y^2+2)^{1/2}(\cancel{2}y)=y(y^2+2)^{1/2}$

$(\frac{dx}{dy})^2=y^2(y^2+2)=y^4+2y^2$

$\sqrt{1+(\frac{dx}{dy})^2}=\sqrt{y^4+2y^2+1}=\sqrt{(y^2+1)^2}=y^2+1$ [Perfect square]

$\Longrightarrow S=2\pi\int_1^2 y(y^2+1)\,dy=2\pi\int_1^2(y^3+y)\,dy=2\pi\left[\frac{y^4}{4}+\frac{y^2}{2}\right]_1^2$

$=2\pi\left[(\frac{16}{4}+\frac{4}{2})-(\frac{1}{4}+\frac{1}{2})\right]=2\pi\,(\frac{42}{8})=\frac{21}{2}\pi$ units

---

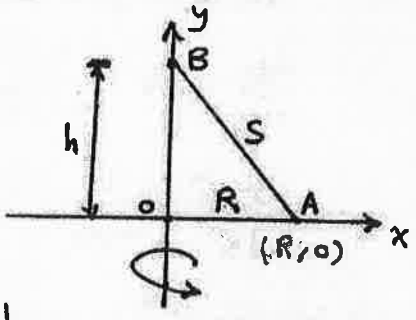
Ex.3: Show that the side surface area of the cone is $A=\pi RS$.

Sol. The point A is $(R,0)$.

To find the point B:

$S^2=h^2+R^2\Rightarrow h=\sqrt{S^2-R^2}$

$\Rightarrow$ the point B is $(0,\sqrt{S^2-R^2})$.



The cone is generated by revolving the line AB about the y-axis.

The side surface area $A=\int_a^b 2\pi r(x)\sqrt{1+(\frac{dy}{dx})^2}\,dx$

$a=0$, $b=R$, $r(x)=x$

Find the equation for the slant line:

$m=\frac{y_2-y_1}{x_2-x_1}=\frac{0-\sqrt{S^2-R^2}}{R-0}=\frac{-\sqrt{S^2-R^2}}{R}$

$y=mx+b\Longrightarrow y=\frac{-\sqrt{S^2-R^2}}{R}x+\sqrt{S^2-R^2}$ ⟵ this step not needed since $\frac{dy}{dx}=m$

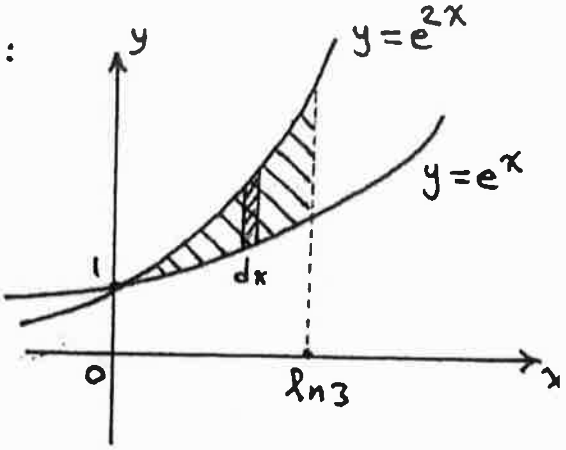
$\frac{dy}{dx}=\frac{-\sqrt{S^2-R^2}}{R}\Longrightarrow(\frac{dy}{dx})^2=\frac{S^2-R^2}{R^2}=\frac{S^2}{R^2}-\frac{R^2}{R^2}=\frac{S^2}{R^2}-1$

$\Longrightarrow A=\int_0^R 2\pi x\sqrt{1+\frac{S^2}{R^2}-1}\,dx=2\pi(\frac{S}{R})\int_0^R x\,dx$

$=2\pi(\frac{S}{R})(\frac{1}{2})[x^2]_0^R=\frac{\pi S}{R}[R^2]=\pi RS$ O.k.

Exercises 6.1 / P.378 :

③ Find the area of the shaded region :



Sol. $A=\int_a^b [f_1(x)-f_2(x)]\,dx$

$a=0$, $b=\ln 3$

$f_1(x)=e^{2x}$, $f_2(x)=e^x$

$\Rightarrow A=\int_0^{\ln 3}(e^{2x}-e^x)\,dx$

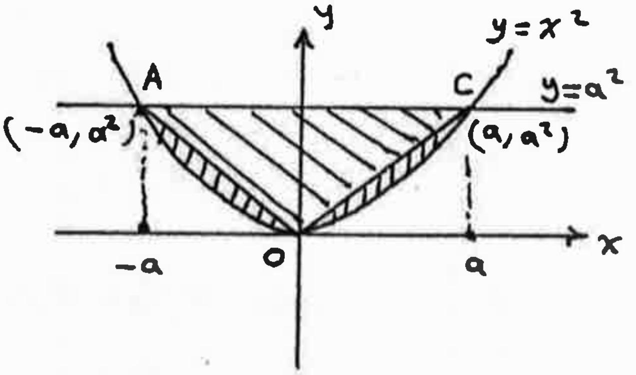
$=[\frac{1}{2}e^{2x}-e^x]_0^{\ln 3}=(\frac{1}{2}e^{2\ln 3}-e^{\ln 3})-(\frac{1}{2}e^0-e^0)$

$=[\frac{1}{2}(9)-3]-[\frac{1}{2}(1)-1]=(\frac{9}{2}-3)-(\frac{1}{2}-1)=2$ units

---

(30) The figure below shows triangle AOC inscribed in the region cut from the parabola $y=x^2$ by the line $y=a^2$. Find the limit of the ratio of the area of the triangle to the area of the parabolic region as a approaches zero.

Sol.



Area of triangle $=\frac{1}{2}(2a)(a^2)$

$=a^3$ units

For the parabolic region :

$A=\int_a^b [f_1(x)-f_2(x)]\,dx$

$a=-a$, $b=a$, $f_1(x)=y_1=a^2$, $f_2(x)=y_2=x^2$

$\Rightarrow A=\int_{-a}^{a}(a^2-x^2)\,dx=[a^2x-\frac{1}{3}x^3]_{-a}^{a}$

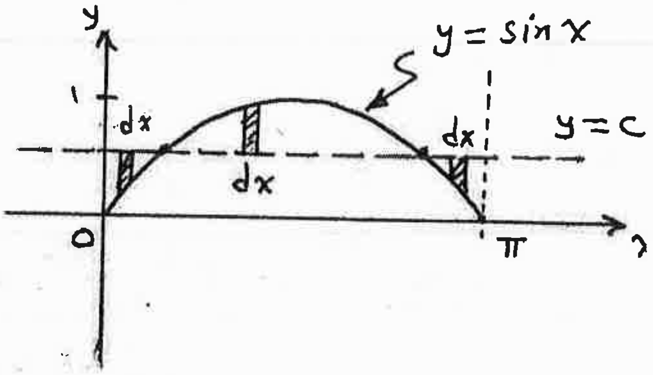
$=[a^2(a)-\frac{1}{3}a^3]-[a^2(-a)-\frac{1}{3}(-a)^3]=[a^3-\frac{1}{3}a^3]-[-a^3+\frac{1}{3}a^3]$

$=a^3-\frac{1}{3}a^3+a^3-\frac{1}{3}a^3=\frac{4}{3}a^3$ units

$\Rightarrow \lim_{a\to 0}\frac{a^3}{\frac{4}{3}a^3}=\lim_{a\to 0}\frac{3}{4}=\frac{3}{4}$

Exercises 6.2 / P. 389 :

③⑨ The curve $y = \sin x$, $0 \leqslant x \leqslant \pi$, is revolved about the line $y = c$ to generate a solid. Find the value of c that minimizes the volume.



Sol. $V = \int_a^b \pi (r(x))^2 \, dx$

$a = 0$, $b = \pi$

$r(x) = |f_1(x) - f_2(x)|$

$= |c - \sin x|$

$$\Rightarrow V = \pi \int_0^{\pi} (c - \sin x)^2 \, dx$$

$$= \pi \int_0^{\pi} (c^2 - 2c \sin x + \sin^2 x) dx \qquad \left\{ \sin^2 x = \frac{1 - \cos 2x}{2} \right.$$

$$= \pi \left[c^2 x + 2c . \cos x + \frac{1}{2} x - \frac{1}{4} \sin 2x \right]_0^{\pi}$$

$$= \pi \left[\left(c^2 \pi + 2c . \cos \pi + \frac{\pi}{2} - \frac{1}{4} \sin 2\pi \right) - \left(0 + 2c . \cos 0 + 0 - \frac{\sin 0}{4} \right) \right]$$

$$= \pi \left[\left(c^2 \pi - 2c + \frac{\pi}{2} \right) - (2c) \right] = \pi \left(c^2 \pi + \frac{\pi}{2} - 4c \right)$$

$$\Rightarrow \boxed{V = \pi \left(c^2 \pi + \frac{\pi}{2} - 4c \right)} \qquad 0 \leqslant c \leqslant 1$$

$$\frac{dV}{dc} = \pi (2c\pi - 4)$$

$$\frac{dV}{dc} = 0 \Rightarrow \pi (2\pi c - 4) = 0 \Rightarrow 2\pi c = 4 \Rightarrow c = \frac{2}{\pi}$$

Check for min. point :

$$\frac{d^2 V}{dc^2} = \pi (2\pi) = 2\pi^2 \quad (+ve) \Rightarrow \smile \quad \text{min. } \underline{\text{o.k.}}$$

at $c = \frac{2}{\pi}$ $\Rightarrow V = \pi \left(\left(\frac{2}{\pi} \right)^2 * \pi + \frac{\pi}{2} - 4 \left(\frac{2}{\pi} \right) \right) = \frac{\pi^2}{2} - 4 \simeq 0.93$

Check bounds :

at $c = 0 \Rightarrow V = \pi \left(0 + \frac{\pi}{2} - 0 \right) = \frac{\pi^2}{2} \simeq 4.93$

at $c = 1 \Rightarrow V = \pi \left(\pi + \frac{\pi}{2} - 4 \right) = \frac{\pi^2}{2} - (4 - \pi)\pi \simeq 2.23$ $\underline{\text{o.k.}}$

$\Rightarrow c = \frac{2}{\pi}$ gives minimum volume.

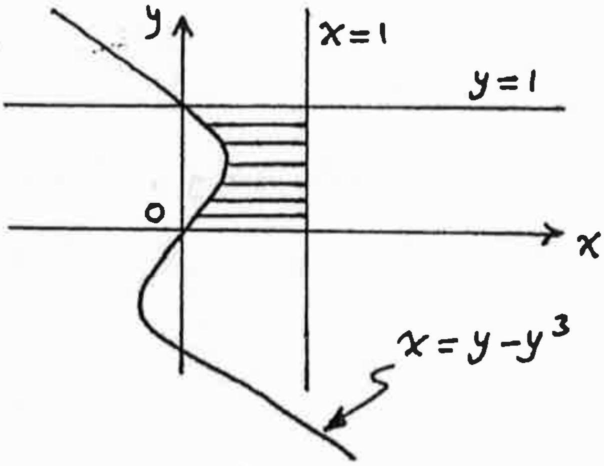
Exercises 6.3 / P. 397

⑲ The region in the first quadrant bounded by $x = y - y^3$, $x = 1$, and $y = 1$ is revolved about:

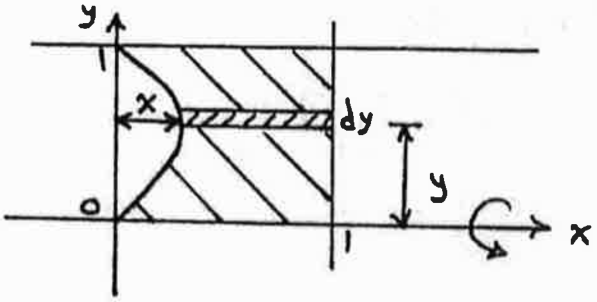
(a) the x-axis (b) the y-axis (c) the line $x = 1$ (d) the line $y = 1$.

Find the volumes generated by each revolving.

Sol.:



(a) About the x-axis:



Use cylindrical shell method:

$$V = \int_c^d 2\pi \cdot r(y) \cdot h(y) \cdot dy$$

$c = 0$, $d = 1$

$r(y) = y$

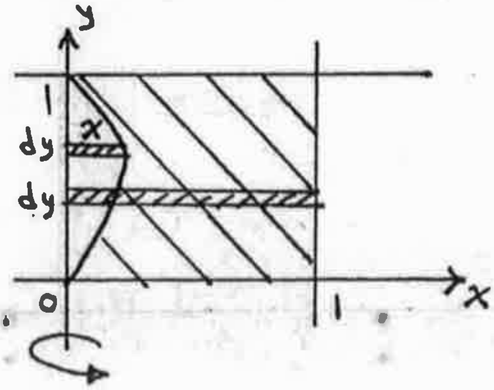
$h(y) = 1 - x = 1 - (y - y^3) = 1 - y + y^3$

$r(y) * h(y) = y(1 - y + y^3) = y - y^2 + y^4$

$$\Rightarrow V = \int_0^1 2\pi (y - y^2 + y^4)\, dy = 2\pi \left[\tfrac{1}{2}y^2 - \tfrac{1}{3}y^3 + \tfrac{1}{5}y^5\right]_0^1$$

$$= 2\pi \left[\tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{5}\right] = \frac{11}{15}\pi \text{ cubic units.}$$

(b) About the y-axis: Use washer method:



$$V = \int_c^d \pi \left[(R(y))^2 - (r(y))^2\right] dy$$

$c = 0$, $d = 1$

$R(y) = 1$, $r(y) = x = y - y^3$

$(R(y))^2 - (r(y))^2 = 1^2 - (y - y^3)^2 = 1 - y^2 + 2y^4 - y^6$

$$\Rightarrow V = \pi \int_0^1 (1 - y^2 + 2y^4 - y^6)\, dy$$

$$= \pi \left[y - \tfrac{1}{3}y^3 + \tfrac{2}{5}y^5 - \tfrac{1}{7}y^7\right]_0^1 = \pi \left[1 - \tfrac{1}{3} + \tfrac{2}{5} - \tfrac{1}{7}\right] = \frac{97}{105}\pi \text{ cubi unit}$$

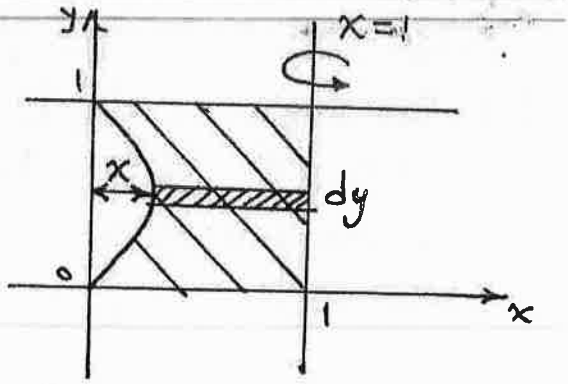
(c) About the line $x = 1$:

Use the disk method :

$$V = \int_c^d \pi \,(r(y))^2 \cdot dy$$

$c = 0$, $d = 1$.

$r(y) = 1 - x = 1 - (y - y^3) = 1 - y + y^3$



$$\Rightarrow V = \int_0^1 \pi (1 - y + y^3)^2 dy$$

$$= \int_0^1 \pi (1 - 2y + y^2 + 2y^3 - 2y^4 + y^6)\, dy$$

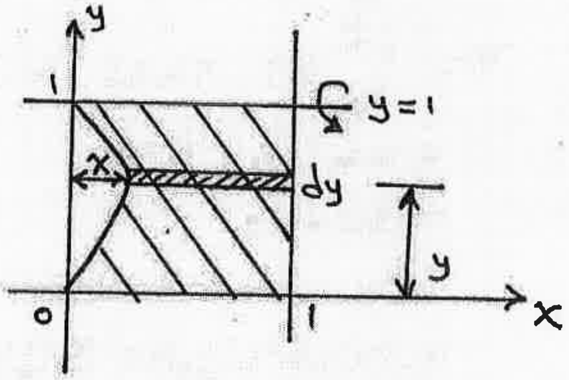
$$= \pi \left[y - y^2 + \frac{1}{3} y^3 + \frac{1}{2} y^4 - \frac{2}{5} y^5 + \frac{1}{7} y^7 \right]_0^1$$

$$= \pi \left[1 - 1 + \frac{1}{3} + \frac{1}{2} - \frac{2}{5} + \frac{1}{7} \right] = \frac{121}{210} \pi$$ cubic units.

(d) About the line $y = 1$:

Use the cylindrical shell method :

$$V = \int_c^d 2\pi \cdot r(y) \cdot h(y) \cdot dy$$



$c = 0$, $d = 1$

$r(y) = 1 - y$

$h(y) = 1 - x = 1 - (y - y^3) = 1 - y + y^3$

$r(y) * h(y) = (1 - y)(1 - y + y^3) = 1 - 2y + y^2 + y^3 - y^4$

$$\Rightarrow V = 2\pi \int_0^1 (1 - 2y + y^2 + y^3 - y^4)\, dy$$

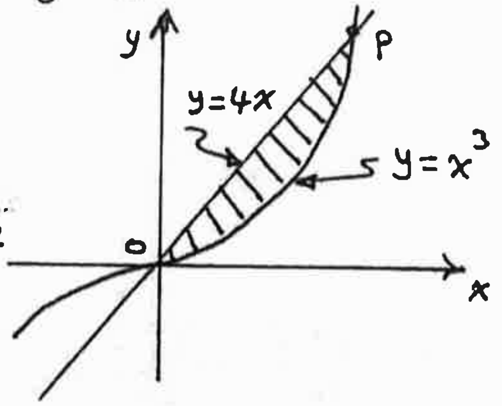
$$= 2\pi \left[y - y^2 + \frac{1}{3} y^3 + \frac{1}{4} y^4 - \frac{1}{5} y^5 \right]_0^1$$

$$= 2\pi \left[1 - 1 + \frac{1}{3} + \frac{1}{4} - \frac{1}{5} \right]$$

$$= \frac{23}{30} \pi$$ cubic units

(21) Find the volume of the solid generated by revolving the region in the first quadrant bounded by $y=x^3$ and $y=4x$ about : (a) the x-axis (b) the line $y=8$.

Sol. First find the point P :



$y_1 = y_2$

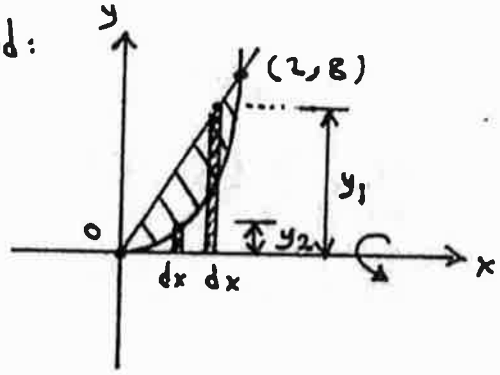
$4x = x^3 \Rightarrow x^3-4x=0 \Rightarrow x(x^2-4)=0$

$\Rightarrow x=0$ and $x^2=4 \Rightarrow x=-2$ & $x=2$

at $x=2 \Rightarrow y=4(2)=8$

$\Longrightarrow$ the point p is $(2,8)$.

(a) About the x-axis: Use washer method:



$$V = \int_a^b \pi\,[(R(x))^2-(r(x))^2]\,dx$$

$a=0$, $b=2$

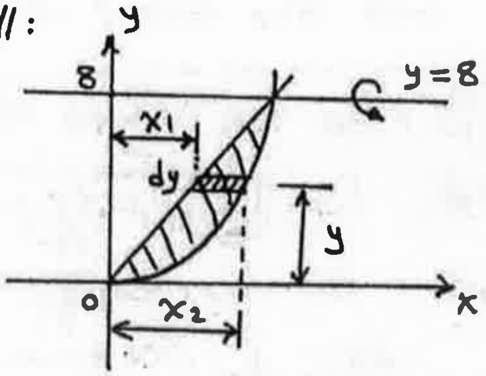
$R(x)=y_1=4x$, $r(x)=y_2=x^3$

$$\Rightarrow V = \int_0^2 \pi\,[(4x)^2-(x^3)^2]\,dx$$

$$= \pi\int_0^2 (16x^2-x^6)\,dx = \pi\left[\frac{16}{3}x^3-\frac{1}{7}x^7\right]_0^2$$

$$= \pi\left[\frac{16}{3}*8-\frac{1}{7}*128\right] = \frac{512}{21}\pi \text{ cubic units.}$$

(b) About the line $y=8$: Use cylindrical shell:



$$V = \int_c^d 2\pi\cdot r(y)\cdot h(y)\cdot dy$$

$c=0$, $d=8$

$r(y) = 8-y$

$h(y) = x_2 - x_1$

$y=x^3 \Rightarrow x = y^{1/3} \Rightarrow x_2 = y^{1/3}$

$y=4x \Rightarrow x = \frac{1}{4}y \Rightarrow x_1 = \frac{1}{4}y$

$\Longrightarrow h(y) = y^{1/3}-\frac{1}{4}y$

$r(y) * h(y) = (8-y)(y^{1/3}-\frac{1}{4}y) = 8y^{1/3}-2y-y^{4/3}+\frac{1}{4}y^2$

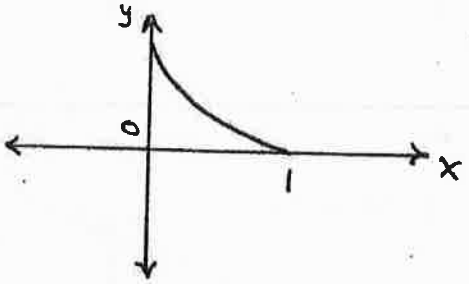
$$\Rightarrow V = \int_0^8 2\pi\left[8y^{1/3}-2y-y^{4/3}+\frac{1}{4}y^2\right]dy = 2\pi\left[\frac{8*3}{4}y^{4/3}-y^2-\frac{3}{7}y^{7/3}+\frac{y^3}{12}\right]_c^1$$

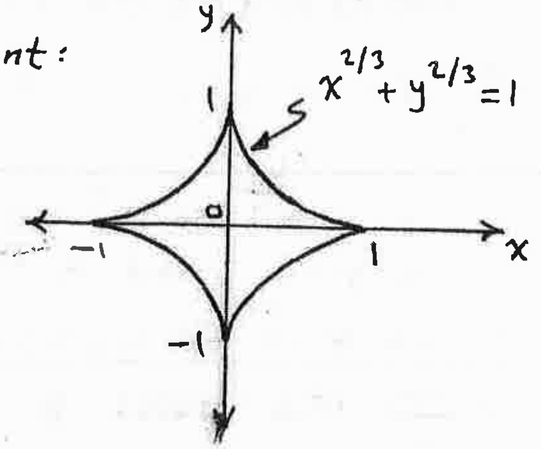
$$= \frac{832}{21}\pi \text{ cubic units}$$

Exercises 6.4 / P.404 :

⑪ Find the length of the astroid $x^{2/3} + y^{2/3} = 1$.

Sol. Take the portion in the 1st quadrant:





$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$a = 0 \quad , b = 1$

$$x^{2/3} + y^{2/3} = 1 \Rightarrow y^{2/3} = 1 - x^{2/3} \Rightarrow y = (1 - x^{2/3})^{3/2}$$

$$\frac{dy}{dx} = \frac{3}{2}(1-x^{2/3})^{1/2}\left(-\frac{2}{3}x^{-1/3}\right) = (1-x^{2/3})^{1/2}(-x^{-1/3})$$

$$\left(\frac{dy}{dx}\right)^2 = (1-x^{2/3})(x^{-2/3}) = x^{-2/3} - x^0 = x^{-2/3} - 1$$

⊛ improper integra

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+x^{-2/3}-1} = (x^{-2/3})^{1/2} = x^{-1/3}$$

$$\Rightarrow L = \boxed{\int_0^1 x^{-1/3}\, dx} \overset{⊛}{=} \frac{3}{2}\left[x^{2/3}\right]_0^1 = \frac{3}{2}[1-0] = \frac{3}{2}$$

$\Longrightarrow$ the total length for the astroid $= 4 * \frac{3}{2} = 6$ units.

---

⑫ Find the length of the curve $y = \int_0^x \sqrt{\cos 2t}\, dt$, $0 \leq x \leq \frac{\pi}{4}$.

Sol. $L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$a = 0 \quad , \quad b = \frac{\pi}{4}$

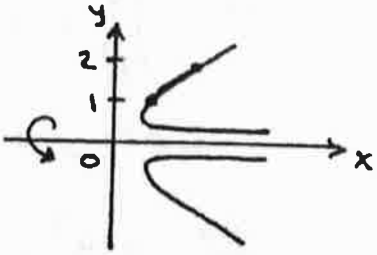
$$\frac{dy}{dx} = \frac{d}{dx}\int_0^x \sqrt{\cos 2t}\, dt = \sqrt{\cos 2x}$$ (1st fundamental theorem)

$$\left(\frac{dy}{dx}\right)^2 = \cos 2x \Longrightarrow \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\cos 2x} = \sqrt{2\cos^2 x} = \sqrt{2}\cos x$$

$$\Rightarrow L = \int_0^{\pi/4} \sqrt{2}\cos x\, dx = \sqrt{2}\left[\sin x\right]_0^{\pi/4} = \sqrt{2}\left[\sin\frac{\pi}{4} - \sin 0\right]$$

$$= \sqrt{2}\left[\frac{1}{\sqrt{2}} - 0\right] = 1 \text{ unit} .$$

⑬ Find the surface area generated by revolving $x=\frac{y^4}{4}+\frac{1}{8y^2}$ from $y=1$ to $y=2$ about the x-axis.
(Hint : Express $dl=\sqrt{(dx)^2+(dy)^2}$ in terms of dy and apply $S=\int 2\pi\rho\, dl$).



Sol. $x=\frac{y^4}{4}+\frac{1}{8y^2}=\frac{1}{4}y^4+\frac{1}{8}y^{-2}$

$$\frac{dx}{dy}=\frac{1}{4}*4*y^3-\frac{2}{8}*y^{-3}=y^3-\frac{1}{4y^3}$$

$$\Rightarrow dx=\left(y^3-\frac{1}{4y^3}\right)dy \Rightarrow (dx)^2=\left(y^3-\frac{1}{4y^3}\right)^2(dy)^2$$
$$=\left(y^6-\frac{1}{2}+\frac{1}{16y^6}\right)(dy)^2$$

$$\Rightarrow dl=\sqrt{(dx)^2+(dy)^2}=\sqrt{\left(y^6-\frac{1}{2}+\frac{1}{16y^6}\right)(dy)^2+(dy)^2}$$
$$=\sqrt{\left(y^6-\frac{1}{2}+\frac{1}{16y^6}\right)+1}\;dy=\sqrt{y^6+\frac{1}{2}+\frac{1}{16y^6}}\;dy$$
$$=\sqrt{\left(y^3+\frac{1}{4y^3}\right)^2}\,dy=\left(y^3+\frac{1}{4y^3}\right)dy \qquad \{\text{perfect square}\}$$

Now $S=\int 2\pi\rho\, dl=\int_c^d 2\pi\cdot r(y)\cdot dl$

$c=1$, $d=2$, $r(y)=y$

$$\Rightarrow S=\int_1^2 2\pi\cdot y\cdot\left(y^3+\frac{1}{4y^3}\right)dy$$
$$=2\pi\int_1^2\left(y^4+\frac{1}{4}y^{-2}\right)dy$$
$$=2\pi\left[\frac{y^5}{5}-\frac{1}{4y}\right]_1^2=2\pi\left[\left(\frac{2^5}{5}-\frac{1}{4(2)}\right)-\left(\frac{1^5}{5}-\frac{1}{4(1)}\right)\right]$$
$$=2\pi\left[\frac{253}{40}\right]=\frac{253}{20}\pi \text{ units}$$

>>>

Note: We can use the eq.② / P.14 (lectures) to solve this problem:

$$S = \int_c^d 2\pi \cdot r(y) \cdot \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$c = 1$, $d = 2$, $r(y) = y$

$$\frac{dx}{dy} = y^3 - \frac{1}{4y^3}$$

$$\left(\frac{dx}{dy}\right)^2 = y^6 - \frac{1}{2} + \frac{1}{16y^6}$$

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{y^6 + \frac{1}{2} + \frac{1}{16y^6}} = \sqrt{\left(y^3 + \frac{1}{4y^3}\right)^2} = y^3 + \frac{1}{4y^3}$$

$$\Rightarrow S = 2\pi \int_1^2 y\left(y^3 + \frac{1}{4y^3}\right) dy$$

$$= 2\pi \int_1^2 \left(y^4 + \frac{1}{4}y^{-2}\right) dy \qquad \{\text{the same integral}\}$$

$$= \frac{253}{20}\pi \text{ cubic units.}$$

CHAPTER SEVEN

THE CALCULUS OF TRANSCENDENTAL FUNCTIONS

Transcendental functions are any functions that are not algebric (i.e. cannot expressed in terms of algebra).
We can say that these functions are transcend the algebri rules.

The transcendental functions are :

1- Trigonometric functions.
2- Inverse trigonometric functions.
3- Logarithmic functions.
4- Exponential functions.

7.1 : Inverse Functions :

First we define the one-to-one function :
One-to-one function is the function that gives one outpu from one input.

For example : $y=2x$ one-to-one.
$y=\sqrt{x}$ one-to-one.
but $y=x^2$ <u>not</u> one-to-one.
$y=\cos x$ <u>not</u> one-to-one.

Inverse function is the function defined by reversing the one-to-one function. The symbol for the inverse function is f^{-1}

$$x \longrightarrow \boxed{f} \longrightarrow y \longrightarrow \boxed{f^{-1}} \longrightarrow x$$

If we have the functions $f(x)$ and $g(x)$:
If $f(g(x))$ and $g(f(x))$ are both identity (or $f(g(x))=g(f(x))=x$; then $f(x)$ and $g(x)$ are inverses of one another (or $f(x)=g^{-1}(x$ and $g(x)=f^{-1}(x)$).

Ex. Given $f(x) = 4x+3$, $g(x) = \frac{x-3}{4}$:

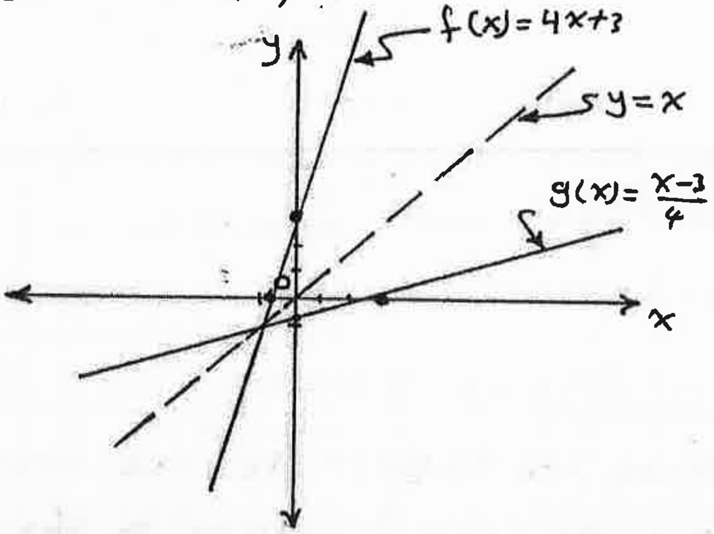
$$f(g(x)) = f\left(\frac{x-3}{4}\right) = 4\left(\frac{x-3}{4}\right)+3 = x$$

$$g(f(x)) = g(4x+3) = \frac{(4x+3)-3}{4} = x$$

$$\Rightarrow f(g(x)) = g(f(x)) = x$$

$$\Rightarrow f(x) = g^{-1}(x) \text{ and } g(x) = f^{-1}(x) .$$

Notice that $f(x)$ and $g(x)$ are reflected about the line $y=x$.



How to find the inverse of a function $y=f(x)$?

1- Solve $y=f(x)$ for x in terms of y (or $x=f(y)$).

2- Interchange x and y.

The resulting formula will be $y=f^{-1}(x)$.

Ex. Find the inverse of $y=\frac{1}{2}x+1$.

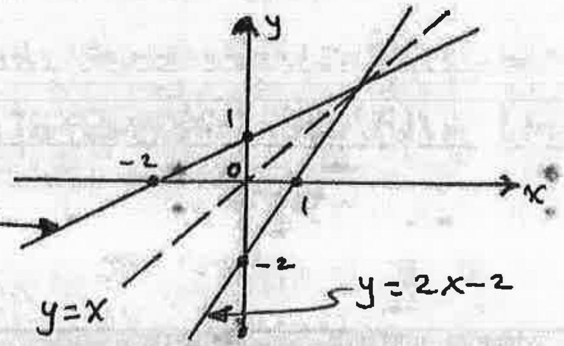
Sol. step1: $y=\frac{1}{2}x+1 \Rightarrow \frac{1}{2}x = y-1 \Rightarrow x = 2y-2$

step2 : $y=2x-2$ is the inverse of $y=\frac{1}{2}x+1$.

For checking : $f(f^{-1}(x)) = \frac{1}{2}(2x-2)+1 = x$

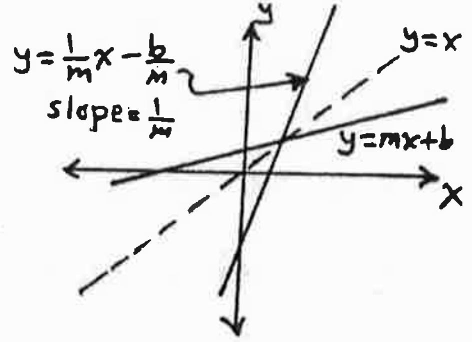
$f^{-1}(f(x)) = 2(\frac{1}{2}x+1)-2 = x$ o.k.

Notice the graph :



Derivative of inverse functions :

Reflecting any nonhorizontal or nonvertical line across the line $y=x$ always inverts the slope of the line.



If the slope of $y=f(x)$ at the point $(a, f(a))$ is $f'(a)$, then the slope of $y=f^{-1}(x)$ at the point $(f(a), a)$ is the reciprocal $\frac{1}{f'(a)}$.

If $b=f(a)$, then : $\boxed{(f^{-1})'(b)=\frac{1}{f'(a)}}$

Ex.

$$f(x)=\frac{1}{2}x+1 \Rightarrow \frac{d}{dx}f(x)=\frac{1}{2}$$
$$f^{-1}(x)=2x-2 \Rightarrow \frac{d}{dx}f^{-1}(x)=2$$

} Notice that $\frac{d}{dx}f(x)=\frac{1}{\frac{d}{dx}f^{-1}(}$

Exercises 7.1/P.456 :

⑥ Given $f(x)=x^2$, $x\leqslant 0$,

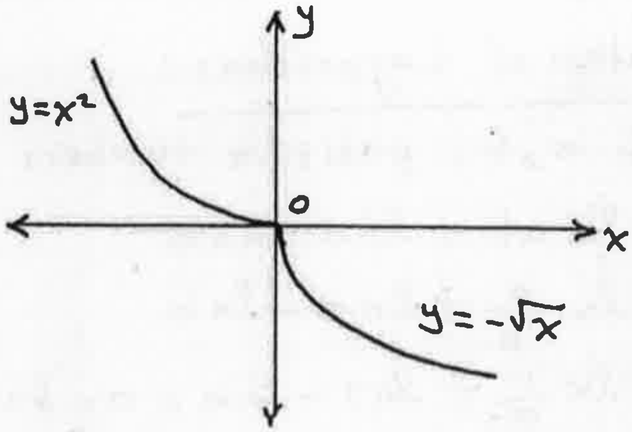
ⓐ Find f^{-1} ⓑ Graph f and f^{-1} together.

Sol. ⓐ $y=x^2 \Rightarrow \sqrt{y}=\sqrt{x^2} \Rightarrow \sqrt{y}=\mp x$

$x\leqslant 0 \Rightarrow \sqrt{y}=-x \Rightarrow x=-\sqrt{y}$

interchange $\Rightarrow y=-\sqrt{x}$ is f^{-1}

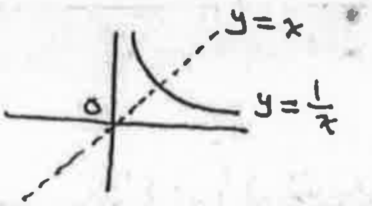
ⓑ The graph:



⑱ Find the inverse of $y=\frac{1}{x}$, $x>0$.

Sol. $y=\frac{1}{x} \Rightarrow x=\frac{1}{y}$

interchange $\Rightarrow y=\frac{1}{x}$ is the inverse.



7.2 : The functions $y = \ln x$ and $y = e^x$:

① The function $y = \ln x$:

The word "logarithm" means power or exponent.
For example $\log_{10} 1000$ means the exponent to which we have to raise to get 1000 (or $\log_{10} 1000 = 3$).

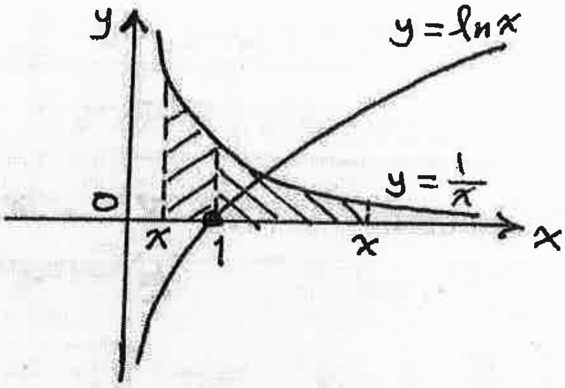
$\log_{10}$ called common logarithm (written log).

$\log_e$ called natural logarithm (written ln).

The natural logarithm function is $y = \ln x$.

$\ln x$ is defined as:

$$\ln x = \int_1^x \frac{1}{t}\,dt \quad , x > 0$$



Notice that :

at $x = 1 \Rightarrow \ln 1 = 0$

for $x > 1 \Rightarrow \ln x$ is +ve.

for $0 < x < 1 \Rightarrow \ln x$ is −ve.

$\lim_{x \to \infty} \ln x = \infty$ & $\lim_{x \to 0^+} \ln x = -\infty$

Rules of Logarithms :

If a, b : positive numbers and n : exponent.

① $\ln ab = \ln a + \ln b$

② $\ln \frac{a}{b} = \ln a - \ln b$

③ $\ln \frac{1}{a} = \ln 1 - \ln a = 0 - \ln a = -\ln a$

④ $\ln a^n = n \ln a$

⑤ $\ln a^{\frac{1}{n}} = \frac{1}{n} \ln a$

Examples :

① $\ln xy = \ln x + \ln y$

② $\ln \frac{2}{3} = \ln 2 - \ln 3$

③ $\ln \frac{1}{5} = -\ln 5$

④ $\ln x^3 = 3 \ln x$

⑤ $\ln \sqrt[3]{y} = \frac{1}{3} \ln y$

The derivative of $y = \ln x$:

$$\ln x = \int_1^x \frac{1}{t}\, dt$$

$$\frac{d}{dx} \ln x = \frac{d}{dx} \int_1^x \frac{1}{t} dt = \frac{1}{x} \Rightarrow \frac{d}{dx} \ln x = \frac{1}{x} \quad , x > 0$$

If $u = f(x)$ and $y = \ln u$:

$$\frac{d}{dx} \ln u = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{d}{du} \ln u \cdot \frac{du}{dx} = \frac{1}{u} \cdot \frac{du}{dx}$$

$$\Rightarrow \boxed{\frac{d}{dx} \ln u = \frac{1}{u} \cdot \frac{du}{dx}} \quad , u > 0$$

The derivative of ln u leads to the integral formula :

$$\boxed{\int \frac{1}{u} du = \ln |u| + c}$$

Ex.1 : If $y = \ln(x^3 + 5x)$, find y' .

Sol. $y' = \frac{1}{x^3 + 5x} \cdot (3x^2 + 5) = \frac{3x^2+5}{x^3+5x}$

Ex.2 : Evaluate $\int \frac{x}{x^2+3} dx$.

sol. Let $u = x^2+3 \Rightarrow du = 2x\, dx \Rightarrow x\, dx = \frac{du}{2}$.

$$\rightarrow \int \frac{1}{u} \cdot \frac{du}{2} = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln |u| + c$$

$$= \frac{1}{2} \ln |x^2+3| + C$$

Note: Sometimes we need (ln) to find the derivative of functions that involve products, quotients, and powers quickly.

Ex. If $y = \frac{(x^2+1)(x+3)^{1/2}}{x-1}$, find $\frac{dy}{dx}$.

sol. Take ln for both sides:

$$\ln y = \ln\left(\frac{(x^2+1)(x+3)^{1/2}}{x-1}\right)$$

$$\ln y = \ln (x^2+1)(x+3)^{1/2} - \ln(x-1)$$

$$\ln y = \ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1)$$

differentiate implicity:

$$\frac{1}{y}\frac{dy}{dx} = \frac{2x}{x^2+1} + \frac{1}{2(x+3)} - \frac{1}{x-1}$$

$$\Longrightarrow \frac{dy}{dx} = y\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right)$$

Substitute for y:

$$\Longrightarrow \frac{dy}{dx} = \frac{(x^2+1)(x+3)^{1/2}}{x-1}\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right)$$

The integral of tan u and cot u:

① $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{du}{u} = -\ln|u| + c = -\ln|\cos x| + c$

$= \ln|\cos x|^{-1} + c = \ln\left|\frac{1}{\cos x}\right| + c = \ln|\sec x| + c$

② $\int \cot x \, dx = \int \frac{\cos x}{\sin x} dx = \int \frac{du}{u} = \ln|u| + c = \ln|\sin x| + c$

The general formulas are:

$$\boxed{\int \tan u \, du = -\ln|\cos u| + c = \ln|\sec u| + c}$$

$$\boxed{\int \cot u \, du = \ln|\sin u| + c}$$

Ex. $\int \tan 2x \, dx$

Sol. Let $u = 2x \Rightarrow du = 2dx \Rightarrow dx = \frac{du}{2}$

$\rightarrow \int \tan u \cdot \frac{du}{2} = \frac{1}{2} \int \tan u \, du = \frac{1}{2} \ln|\sec u| + c$

$= \frac{1}{2} \ln|\sec 2x| + C$

---

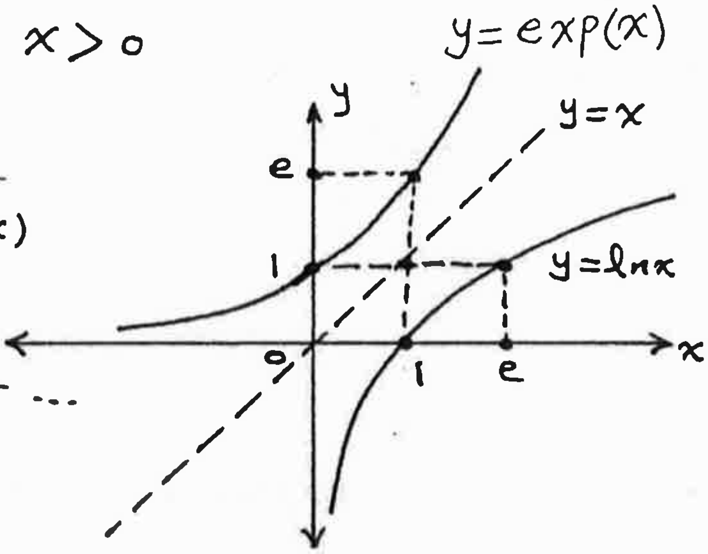
2) The Exponential Function $y = \exp(x)$:

The function $y = \exp(x)$ is the inverse of $y = \ln x$.

Then $\ln(\exp(x)) = x$, for all x.

$\exp(\ln(x)) = x$, $x > 0$

The graph of $y = \exp(x)$:



The number e on $y = \exp(x)$ is the number whose $x = 1$.

$e = 2.7\,1828\,1828\,45\,90\,45 \ldots$

Notice that $\ln e = 1$

$$\lim_{x \to \infty} \exp(x) = \infty$$

$$\lim_{x \to -\infty} \exp(x) = 0$$

The function $y = e^x$:

For e^x : take $\ln \Rightarrow \ln e^x = x \ln e = x(1) = x$

but $\ln(\exp(x)) = x$

$\Rightarrow \boxed{\exp(x) = e^x}$ for all x.

Now: From $\exp(\ln x) = x \Rightarrow \boxed{e^{\ln x} = x}$

From $\ln(\exp(x)) = x \Rightarrow \boxed{\ln e^x = x}$

Rules of Exponents :

① $e^{x} = \frac{1}{e^{-x}}$ for all x.

② $e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2}$

③ $\frac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2}$

Examples :

① $\ln e^{2} = 2 \ln e = 2$

② $e^{\ln 2} = 2$

③ $\ln \sqrt{e} = \ln e^{1/2} = \frac{1}{2}$

④ $e^{3 \ln 2} = e^{\ln 2^{3}} = 2^{3} = 8$

⑤ $\ln e^{\cos x} = \cos x$

⑥ $e^{\ln(x^{2}+1)} = x^{2}+1$

⑦ $e^{3} \cdot e^{4} = e^{3+4} = e^{7}$

⑧ $\frac{e^{x}}{e^{-2}} = e^{x-(-2)} = e^{x+2}$

Useful Rules :

① To remove logarithms from an equation exponentiate both sides.

② To remove exponents from an equation take logarithm for both sides.

Ex.1 : Given $\ln y = 3x+4$. Find y.

sol. exponentiate both sides $\Rightarrow e^{\ln y} = e^{3x+4}$

$\Rightarrow y = e^{3x+4}$

Ex.2 : Given $\ln(y-2) = \ln(\sin x) - x$. Find y.

sol. $\ln(y-2) - \ln(\sin x) = -x \Rightarrow \ln \frac{y-2}{\sin x} = -x$

$\Rightarrow e^{\ln \frac{y-2}{\sin x}} = e^{-x} \Rightarrow \frac{y-2}{\sin x} = e^{-x} \Rightarrow y = e^{-x} \sin x + 2$

Ex.3 : If $e^{3k} = 15$, find k.

sol. Take $\ln \Rightarrow \ln e^{3k} = \ln 15 \Rightarrow 3k = \ln 15 \Rightarrow k = \frac{1}{3} \ln 15$

Derivative of $y=e^x$:

$y=e^x \Rightarrow \ln y = \ln e^x \Rightarrow \ln y = x$

differentiate $\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = y \Rightarrow \frac{d}{dx} e^x = e^x$

If $u=f(x) \Longrightarrow \boxed{\frac{d}{dx} e^u = e^u \cdot \frac{du}{dx}}$

Ex.1 $\frac{d}{dx} e^{\tan x} = e^{\tan x} \cdot \sec^2 x$

Ex.2 $\frac{d}{dx} e^{\sin 3x} = e^{\sin 3x} \cdot \cos 3x \cdot 3$

The integral of e^u :

As $\frac{d}{dx} e^u = e^u \cdot \frac{du}{dx} \Longrightarrow \boxed{\int e^u\, du = e^u + c}$

Ex.1 $\int e^{\sin x} \cdot \cos x\, dx$.

Let $u = \sin x \Rightarrow du = \cos x\, dx$

$\rightarrow \int e^u \cdot du = e^u + c = e^{\sin x} + C$

Ex.2 $\int_0^{\ln 2} e^{3x}\, dx$.

Let $u = 3x \Rightarrow du = 3dx \Rightarrow dx = \frac{du}{3}$

L.L. $= 3(0) = 0$

U.L. $= 3\ln 2 = \ln 2^3 = \ln 8$

$\longrightarrow \int_0^{\ln 8} e^u \cdot \frac{du}{3} = \frac{1}{3}\int_0^{\ln 8} e^u \cdot du = \frac{1}{3}\left[e^u\right]_0^{\ln 8}$

$= \frac{1}{3}\left[e^{\ln 8} - e^0\right] = \frac{1}{3}\left[8-1\right] = \frac{7}{3}$

Examples for $\ln x$ and e^x :

Find dy/dx :

① $y = \ln(\ln x^2)$

$$\frac{dy}{dx} = \frac{1}{\ln x^2} \cdot \frac{1}{x^2} \cdot 2x = \frac{2}{x \ln x^2}$$

② $y = (\ln x)^3$

$$\frac{dy}{dx} = 3(\ln x)^2 \left(\frac{1}{x}\right) = \frac{3 \ln^2 x}{x}$$

③ $y = x\left[\sin(\ln x) + \cos(\ln x)\right]$

$$\frac{dy}{dx} = x\left[(\cos(\ln x))\left(\frac{1}{x}\right) - (\sin(\ln x))\left(\frac{1}{x}\right)\right] + \left[\sin(\ln x) + \cos(\ln x)\right](1$$

$$= 2\cos(\ln x)$$

④ $y = e^{\sqrt{x}}$

$$\frac{dy}{dx} = e^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\right) = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

Evaluate :

① $\displaystyle\int_e^{e^2} \frac{dx}{x \ln x}$

Let $u = \ln x \Rightarrow du = \frac{1}{x} dx$

$U.L. = \ln e^2 = 2 \ln e = 2$

$L.L. = \ln e = 1$

$$\longrightarrow \int_1^2 \frac{du}{u} = \left[\ln|u|\right]_1^2 = \ln 2 - \ln 1 = \ln 2 - 0 = \ln 2$$

② $\displaystyle\int_0^{\pi} \frac{\sin x}{2 - \cos x} dx$

$$= \left[\ln|2 - \cos x|\right]_0^{\pi} = \ln|2 - \cos\pi| - \ln|2 - \cos 0|$$

$$= \ln|2 - (-1)| - \ln|2 - 1|$$

$$= \ln 3 - \ln 1 = \ln 3$$

Exercises 7.2/P.464 :

(23) Evaluate $\int_1^2 \frac{2\ln x}{x}\,dx$

sol. $2\int_1^2 \frac{\ln x}{x}\,dx = 2\left[\frac{\ln^2 x}{2}\right]_1^2 = \left[\ln^2 x\right]_1^2$

$= \ln^2 2 - \ln^2 1 = \ln^2 2.$

(26) $\int_{\ln\frac{\pi}{6}}^{\ln\frac{\pi}{2}} 2e^x \cos(e^x)\,dx$

sol. $2\int_{\ln\frac{\pi}{6}}^{\ln\frac{\pi}{2}} \cos(e^x).e^x\,dx = 2\left[\sin(e^x)\right]_{\ln\frac{\pi}{6}}^{\ln\frac{\pi}{2}}$

$= 2\left[\sin(e^{\ln\frac{\pi}{2}}) - \sin(e^{\ln\frac{\pi}{6}})\right] = 2\left[\sin\frac{\pi}{2} - \sin\frac{\pi}{6}\right]$

$= 2\left[1 - \frac{1}{2}\right] = 2\left(\frac{1}{2}\right) = 1$

(29) $\int_{\pi/2}^{\pi} 2\cot\frac{x}{3}\,dx$

sol. $2*3\int_{\pi/2}^{\pi} \cot\frac{x}{3}\cdot\frac{1}{3}\,dx = 6\left[\ln\left|\sin\frac{x}{3}\right|\right]_{\pi/2}^{\pi}$

$= 6\left[\ln\left|\sin\frac{\pi}{3}\right| - \ln\left|\sin\frac{\pi}{6}\right|\right] = 6\left[\ln\frac{\sqrt{3}}{2} - \ln\frac{1}{2}\right]$

$= 6\left[(\ln\sqrt{3} - \ln 2) - (\ln 1 - \ln 2)\right] = 6\left[\ln\sqrt{3} - \ln 2 + \ln 2\right]$

$= 6\ln\sqrt{3} = 6\ln 3^{1/2} = \ln 3^{6(1/2)} = \ln 3^3 = \ln 27$

Find the limits:

(31) $\lim_{x\to\infty} \ln\frac{1}{x} = \ln\frac{1}{\infty} = \ln 0 = -\infty$

(33) $\lim_{x\to\infty} e^{-x} = \lim_{x\to\infty}\frac{1}{e^x} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0$

(43) Find the maximum value of $f(x) = x^2 \ln \frac{1}{x}$.

sol. $y = x^2 \ln \frac{1}{x} = x^2 \ln x^{-1} = -x^2 \ln x$

$$\frac{dy}{dx} = -x^2\left(\frac{1}{x}\right) + \ln x(-2x) = -x - 2x\ln x$$

$$= -x(1+2\ln x)$$

To find max. value : $\frac{dy}{dx} = 0 \Rightarrow -x(1+2\ln x) = 0$

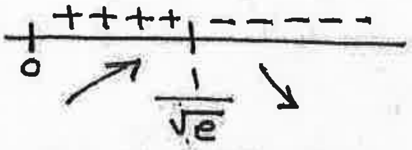
$\Rightarrow -x = 0 \Rightarrow \boxed{x=0}$ and $1 + 2\ln x = 0 \Rightarrow \ln x = \frac{-1}{2}$

$\Rightarrow e^{\ln x} = e^{-1/2} \Rightarrow \boxed{x = e^{-1/2}}$

For $x=0$ the function is not defined $[0 * \ln \frac{1}{0} = 0 * \infty]$

For $x = e^{-1/2} = \frac{1}{\sqrt{e}}$:

From the sign of $\frac{dy}{dx}$



$\Rightarrow$ at $x = \frac{1}{\sqrt{e}}$ the function has local maximum value.

at $x = \frac{1}{\sqrt{e}} \Rightarrow y = -\left(\frac{1}{\sqrt{e}}\right)^2 * \ln e^{-1/2} = -\frac{1}{e} * -\frac{1}{2}\ln e = \frac{1}{2e}$

As the domain of $f(x)$ is $x > 0$ and at $x > \frac{1}{\sqrt{e}}$ the function decreasing then $y = \frac{1}{2e}$ is absolute max. value.

(45) Solve the initial value problem:

$\frac{dy}{dx} = \cos x \cdot e^{\sin x}$, initial conditions: at $x=0$, $y=0$.

sol. $dy = \cos x \cdot e^{\sin x} . dx$

$\Rightarrow \int dy = \int \cos x \cdot e^{\sin x} . dx \Rightarrow y = e^{\sin x} + c$

$x=0, y=0 \Rightarrow 0 = e^{\sin 0} + c \Rightarrow 0 = e^0 + c \Rightarrow 0 = 1 + c$

$\Rightarrow c = -1 \Rightarrow y = e^{\sin x} - 1$.

7.3 : The Functions $y=a^x$ and $y=\log_a x$:

① The function $y=a^x$:

If a is positive number then $a^x = e^{\ln a^x} = e^{x\ln a}$

$\Longrightarrow \boxed{a^x = e^{x\ln a}}$

Ex.1 : $5^{\sqrt{3}} = e^{\sqrt{3}\ln 5}$

Ex.2 : $3^{\pi} = e^{\pi \ln 3}$

Laws of exponents :

If a is positive and x & y are any numbers then :

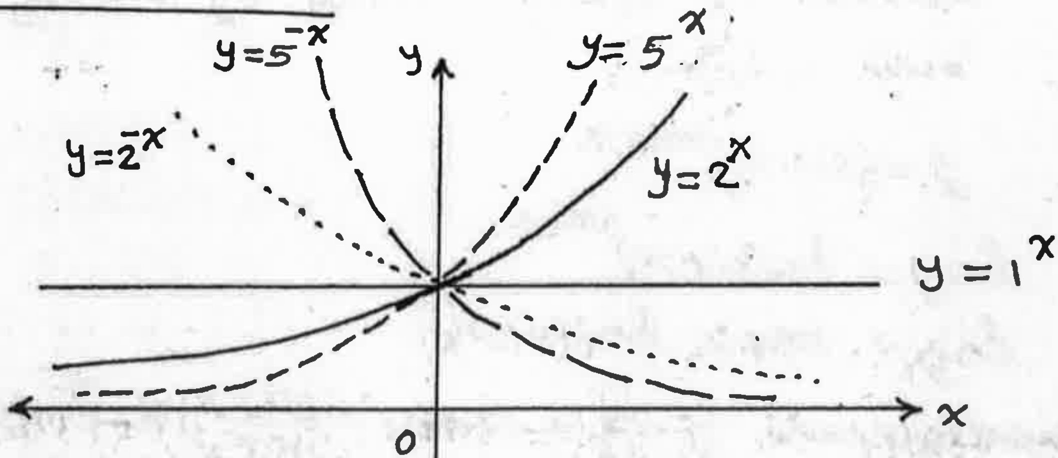
① $a^x \cdot a^y = a^{x+y}$

② $\frac{a^x}{a^y} = a^{x-y}$

③ $a^{-x} = \frac{1}{a^x}$

④ $(a^x)^y = a^{xy} = (a^y)^x$

The graph of $y=a^x$:



Notice that:

$\lim_{x\to\infty} a^x = \infty$ & $\lim_{x\to\infty} a^{-x} = 0$

$\lim_{x\to -\infty} a^x = 0$ & $\lim_{x\to -\infty} a^{-x} = \infty$

We can see that :
the graph of $y=a^{-x}$ comes from reflecting the graph of $y=a^x$ about the y-axis.

The derivative of $y=a^u$:

$$\frac{d}{dx}a^x=\frac{d}{dx}e^{x\ln a}=e^{x\ln a}.\ln a=a^x.\ln a$$

If $u=f(x)\Rightarrow \boxed{\frac{d}{dx}a^u=a^u.\ln a.\frac{du}{dx}}$

Ex. $\frac{d}{dx}5^{\sin 3x}=5^{\sin 3x}.\ln 5.\cos 3x.3$

Note ①: The derivative of $y=a^u$ may be found by taking ln for both sides of $y=a^u$.

Ex. $y=8^x$

$\ln y=\ln 8^x \Rightarrow \ln y=x\ln 8$

differentiate $\Rightarrow \frac{1}{y}\frac{dy}{dx}=\ln 8$

$\Rightarrow \frac{dy}{dx}=y*\ln 8=8^x.\ln 8$

Note ②: For the general power functions ($y=u^u$) the derivative always found by taking ln for both sides.

Ex. $y=(\sin x)^{\tan x}$

$\ln y=\ln(\sin x)^{\tan x}$

$\ln y=\tan x\ln(\sin x)$

differentiate $\Rightarrow \frac{1}{y}\frac{dy}{dx}=\tan x\left(\frac{\cos x}{\sin x}\right)+\ln(\sin x)(\sec^2 x)$

$=1+\sec^2 x\ln(\sin x)$

$\Rightarrow \frac{dy}{dx}=y[1+\sec^2 x\ln(\sin x)]$

$=(\sin x)^{\tan x}[1+\sec^2 x\ln(\sin x)]$

The integral of a^u :

$$\frac{d}{dx}a^u = a^u . \ln a . \frac{du}{dx}$$

$$[\div \ln a] \Rightarrow \frac{1}{\ln a} . \frac{d}{dx} a^u = a^u . \frac{du}{dx}$$

integrate with dx :

$$\Rightarrow \int \frac{1}{\ln a} . \frac{d}{dx} a^u . dx = \int a^u . \frac{du}{dx} . dx$$

$$\Rightarrow \int a^u . du = \frac{1}{\ln a} \int \frac{d}{dx} a^u . dx = \frac{1}{\ln a} . a^u + C$$

$$\Rightarrow \boxed{\int a^u du = \frac{a^u}{\ln a} + c}$$

Ex. $\int 5^{\sin x} . \cos x dx = \frac{1}{\ln 5} 5^{\sin x} + C$

---

② The function $y = \log_a x$:

The function $y = \log_a x$ is the inverse of $y = a^x$.

$\Rightarrow \log_a(a^x) = x$ for all x .

and $a^{(\log_a x)} = x$ for positive x .

Examples:

① $\log_2 2^5 = 5$

② $3^{\log_3 7} = 7$

The evaluation of $\log_a x$:

$$a^{\log_a x} = x$$

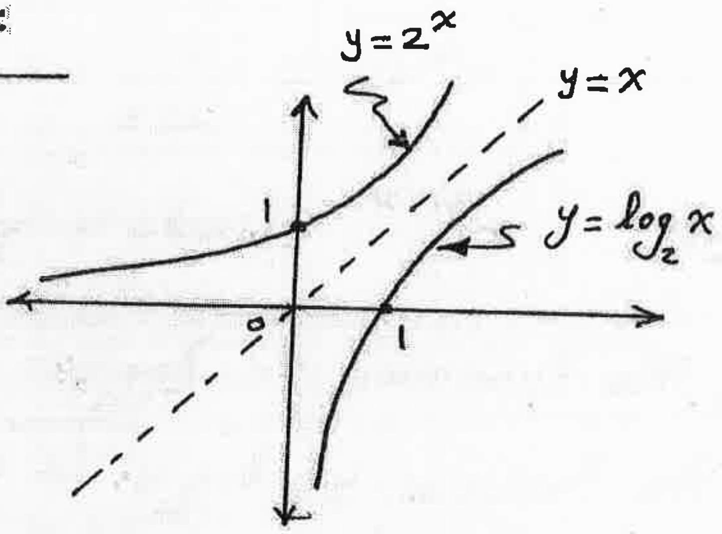
take ln $\Longrightarrow \ln a^{\log_a x} = \ln x \Longrightarrow \log_a x \cdot \ln a = \ln x$

$$\Longrightarrow \boxed{\log_a x = \frac{\ln x}{\ln a}}$$

Ex.① $\log_2 5 = \dfrac{\ln 5}{\ln 2}$ Ex.②: $\log_{10} 2 = \dfrac{\ln 2}{\ln 10}$

The graph of $y = \log_a x$:

$y = \log x$ is obtained by reflecting the graph of $y = a^x$ across the line $y = x$.



The derivative of $\log_a u$:

$$\frac{d}{dx}\log_a u = \frac{d}{dx}\left(\frac{\ln u}{\ln a}\right) = \frac{1}{\ln a}\cdot\frac{d}{dx}\ln u = \frac{1}{\ln a}\cdot\frac{1}{u}\cdot\frac{du}{dx}$$

$$\Longrightarrow \boxed{\frac{d}{dx}\log_a u = \frac{1}{\ln a}\cdot\frac{1}{u}\cdot\frac{du}{dx}}$$

Ex. $\dfrac{d}{dx}\log_{10}(3x+1) = \dfrac{1}{\ln 10}\cdot\dfrac{1}{3x+1}\cdot 3 = \dfrac{3}{(\ln 10)(3x+1)}$

Integrals involving $\log_a x$:

Convert $\log_a x$ to $\dfrac{\ln x}{\ln a}$.

Ex. $$\int \frac{\log_2 x}{x}\,dx = \int \frac{\ln x}{\ln 2}\cdot\frac{dx}{x} = \frac{1}{\ln 2}\int\frac{\ln x}{x}\,dx$$

$$= \frac{1}{\ln 2}\cdot\frac{\ln^2 x}{2} + C = \frac{\ln^2 x}{2\ln 2} + C$$

Exercises 7.3 / P. 472 :

④ Find $\frac{dy}{dx}$ for $y = x^{1-e}$.

sol. $$\frac{dy}{dx} = (1-e)\,x^{(1-e)-1} = \frac{1-e}{x^e}$$

⑪ Find $\frac{dy}{dx}$ for $y = x^{\ln x}$

sol. take ln $\Rightarrow$ $\ln y = \ln x^{\ln x} = \ln x . \ln x = (\ln x)^2$

differentiate : $\dfrac{1}{y}\dfrac{dy}{dx} = 2\ln x\left(\frac{1}{x}\right)$

$$\Rightarrow \frac{dy}{dx} = 2y\,\frac{\ln x}{x} = 2x^{\ln x}\left(\frac{\ln x}{x}\right) = x^{\ln x}\left(\frac{\ln x^2}{x}\right)$$

⑱ Evaluate $\int_1^e x^{\ln 2 - 1}\,dx \longrightarrow \int u^n\,du$

sol. $$= \left[\frac{x^{\ln 2}}{\ln 2}\right]_1^e = \left[\frac{e^{\ln 2}}{\ln 2} - \frac{1^{\ln 2}}{\ln 2}\right] = \frac{2}{\ln 2} - \frac{1}{\ln 2} = \frac{1}{\ln 2}$$

㉕ Evaluate $\int_1^{\sqrt{2}} 2^{x^2}\cdot x\,dx$

sol $$= \frac{1}{2}\left[\frac{2^{x^2}}{\ln 2}\right]_1^{\sqrt{2}} = \frac{1}{2}\left[\frac{2^2}{\ln 2} - \frac{2^1}{\ln 2}\right] = \frac{1}{2}\left[\frac{2}{\ln 2}\right]$$

$$= \frac{1}{\ln 2}$$

㊳ Solve the equation $8^{\log_8 3} - e^{\ln 5} = x^2 - 7^{\log_7 3x}$

<u>sol.</u> $3 - 5 = x^2 - 3x$

$\Longrightarrow x^2 - 3x + 2 = 0$

$(x-2)(x-1) = 0$

$\Longrightarrow x = 2$ and $x = 1$

㊼ Find dy/dx for $y = \dfrac{1}{\log_2 x}$

<u>sol.</u> $y = (\log_2 x)^{-1} \Longrightarrow \dfrac{dy}{dx} = -(\log_2 x)^{-2}\left(\dfrac{1}{x\ln 2}\right)$

$$= \frac{-1}{x\ln 2 \, \log_2^2 x}$$

(53) Evaluate $\displaystyle\int_1^4 \frac{\ln 2 \, \log_2 x}{x}\,dx$

<u>sol.</u> $\displaystyle = \int_1^4 \frac{\ln 2 \cdot \frac{\ln x}{\ln 2}}{x}\,dx = \int_1^4 \frac{\ln x}{x}\,dx = \frac{1}{2}\left[\ln^2 x\right]_1^4$

$= \frac{1}{2}\left[\ln^2 4 - \ln^2 1\right] = \frac{1}{2}\ln^2 4 = \frac{1}{2}(\ln 4)^2 = \frac{1}{2}(\ln 2^2)^2$

$= \frac{1}{2}(2\ln 2)^2 = \frac{1}{2}(4\ln^2 2) = 2\ln^2 2$

(59) Find the limits:

(a) $\displaystyle\lim_{x\to\infty} \log_2 x = \lim_{x\to\infty} \frac{\ln x}{\ln 2} = \frac{1}{\ln 2}\lim_{x\to\infty} \ln x = \frac{1}{\ln 2}(\infty)$

$= \infty$

(b) $\displaystyle\lim_{x\to\infty} \log_2\left(\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\ln\frac{1}{x}}{\ln 2} = \frac{1}{\ln 2}\lim_{x\to\infty} \ln\frac{1}{x}$

$= \frac{1}{\ln 2} \cdot \ln\frac{1}{\infty} = \frac{1}{\ln 2} \cdot \ln 0 = \frac{1}{\ln 2}(-\infty)$

$= -\infty$

7.7: The Inverse Trigonometric Functions :

The inverse trigonometric functions are used to find the angles from the triangle sides. They also provide antiderivatives for a wide variety of functions.

The inverse of sine :

$y = \sin x$ is not one-to-one function, but if we take the interval $\frac{-\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$, the function will be one-to-one and therefore has an inverse which denoted by : $y = \sin^{-1} x$ (or can be written : $y = \arc \sin x$)

Notice that: $y = \sin^{-1} x \iff x = \sin y$.

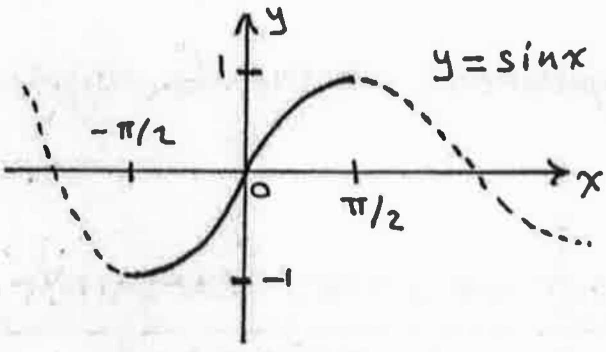
For example : $\frac{\pi}{2} = \sin^{-1} 1 \iff 1 = \sin \frac{\pi}{2}$

Notes : ① $\sin(\sin^{-1} x) = x$ and $\sin^{-1}(\sin x) = x$

② $\sin^{-1} x \neq (\sin x)^{-1}$ where $(\sin x)^{-1} = \frac{1}{\sin x}$

The graph of $y = \sin^{-1} x$:

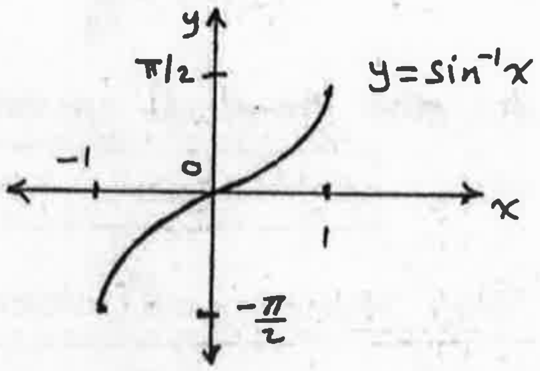
From the graph of $y = \sin x$, take the interval $[\frac{-\pi}{2}, \frac{\pi}{2}]$



$\frac{-\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$

$-1 \leqslant y \leqslant 1$

$\Rightarrow$



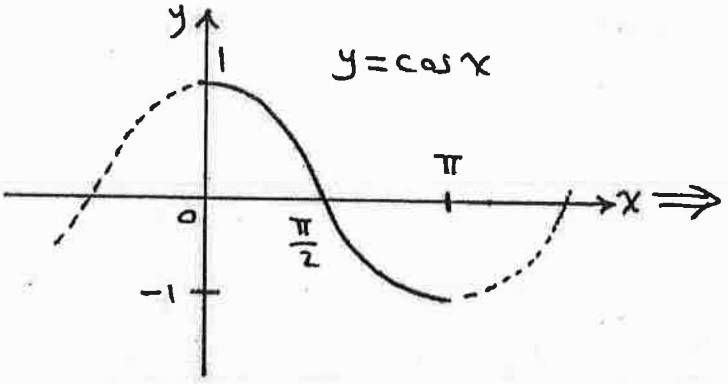
$-1 \leqslant x \leqslant 1$

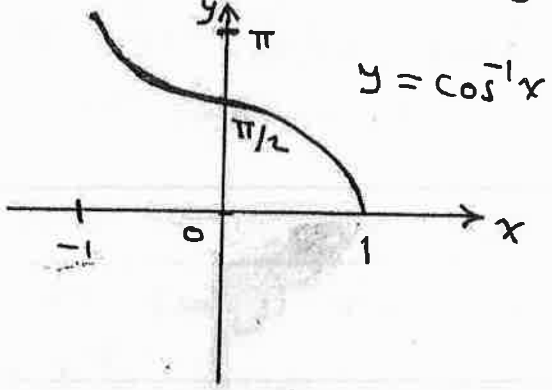
$\frac{-\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$

The graph of $y = \sin^{-1} x$ is symmetric about the origin, then $\boxed{\sin^{-1}(-x) = -\sin^{-1} x}$.

The inverse of cosine:

From the graph of $y=\cos x$, take the interval $[0, \pi]$.

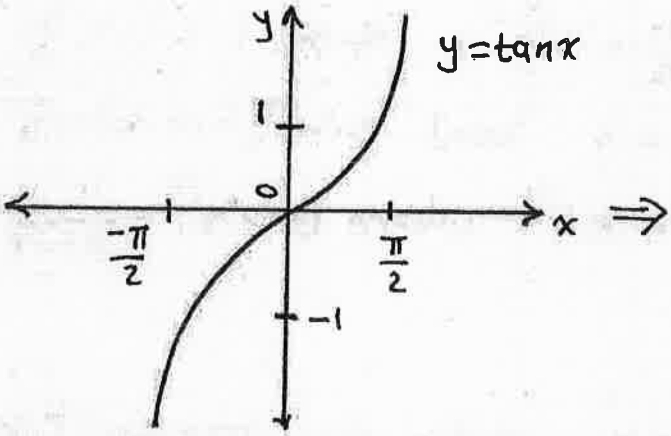


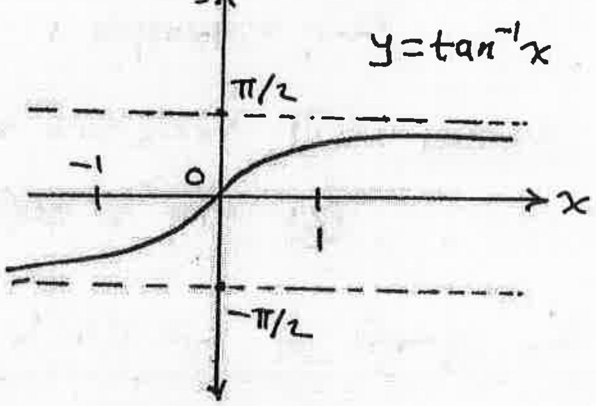


$0 \leqslant x \leqslant \pi$ $\qquad$ $-1 \leqslant x \leqslant 1$

The inverse of tangent:

From the graph of $y=\tan x$, take the interval $(\frac{-\pi}{2}, \frac{\pi}{2})$.





$\frac{-\pi}{2} < x < \frac{\pi}{2}$ $\qquad$ $-\infty < x < \infty$

As the graph of $y=\tan^{-1}x$ is symmetric about the origin, then $\boxed{\tan^{-1}(-x) = -\tan^{-1} x}$

The inverse of secant, cosecant, and Cotangent:

① For $y=\sec^{-1}x$, take the interval $0 \leqslant x \leqslant \pi$, $x \neq \frac{\pi}{2}$ from the graph of $y = \sec x$.

② For $y=\csc^{-1}x$, take $\frac{-\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$, $x \neq 0$ from $y=\csc x$.

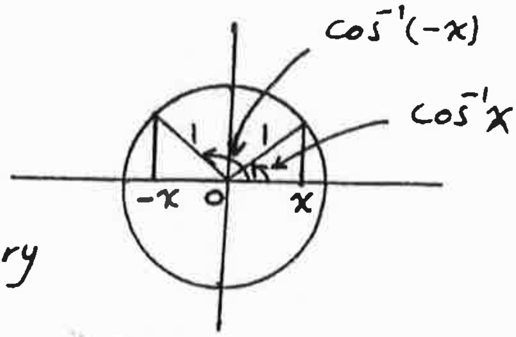
③ For $y=\cot^{-1}x$, take $0 < x < \pi$ from $y=\cot x$.

Notes:

① From the unit circle:

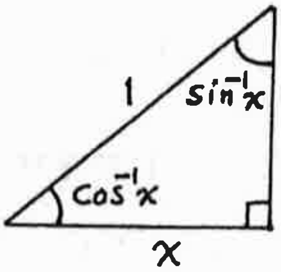
$$\boxed{\cos^{-1}x + \cos^{-1}(-x) = \pi}$$

$\cos^{-1}x$ and $\cos^{-1}(-x)$ are complementary angles.



② From the triangle in unit circle:

$$\boxed{\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}}$$



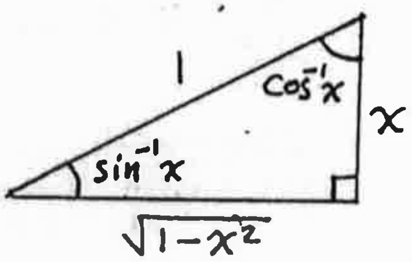
Also:

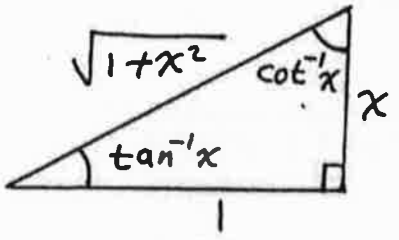
$$\boxed{\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}} \quad \& \quad \boxed{\sec^{-1}x + \csc^{-1}x = \frac{\pi}{2}}$$

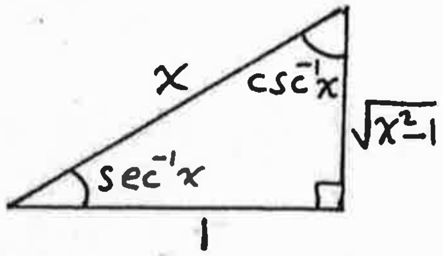
③ $\boxed{\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right)}$, $\boxed{\csc^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)}$, $\boxed{\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)}$

Right - Triangle Interpretation:

These triangles are useful in integration problems.



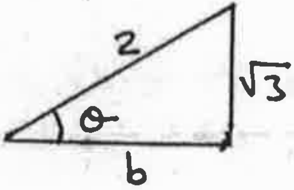




Ex. If $\theta = \sin^{-1}\frac{\sqrt{3}}{2}$, find $\sec\theta$ and $\tan\theta$.

Sol. $\theta = \sin^{-1}\frac{\sqrt{3}}{2} \Longrightarrow \sin\theta = \frac{\sqrt{3}}{2}$

From the reference triangle:



$$b = \sqrt{(2)^2 - (\sqrt{3})^2} = \sqrt{1} = 1$$

$$\Longrightarrow \sec\theta = \frac{2}{1} = 2 \quad \& \quad \tan\theta = \frac{\sqrt{3}}{1} = \sqrt{3}$$

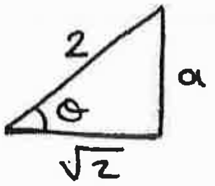
Exercises 7.7/P. 497 :

Evaluate :

(15) $\sin\left(\cos^{-1}\frac{\sqrt{2}}{2}\right)$

sol. $\cos^{-1}\frac{\sqrt{2}}{2} = \theta \Rightarrow \cos\theta = \frac{\sqrt{2}}{2}$

$$a = \sqrt{(2)^2 - (\sqrt{2})^2} = \sqrt{2}$$



$$\Rightarrow \sin\left(\cos^{-1}\frac{\sqrt{2}}{2}\right) = \sin\theta = \frac{a}{2} = \frac{\sqrt{2}}{2}$$

(27) $\sin^{-1}(1) - \sin^{-1}(-1)$

sol. $\sin^{-1}(1) - \sin^{-1}(-1) = \sin^{-1}(1) + \sin^{-1}(1)$ from: $[\sin^{-1}(-x) = -\sin^{-1}(x)]$

$$= 2\sin^{-1}(1) = 2\left(\frac{\pi}{2}\right) = \pi$$

7.8 : Derivatives of Inverse Trigonometric Functions:

The derivative of $y = \sin^{-1}x$:

$$\sin(\sin^{-1}x) = x$$

differentiate both sides: $\Rightarrow \cos(\sin^{-1}x) \cdot \frac{d}{dx}\sin^{-1}x = 1$

$$\Rightarrow \frac{d}{dx}\sin^{-1}x = \frac{1}{\cos(\sin^{-1}x)} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}}$$

but $\sin y = x$ as $y = \sin^{-1}x$

$$\Rightarrow \frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$$

If $u = f(x)$, then:

$$\boxed{\frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}}$$

The derivatives of inverse trigonometric functions are:

① $\frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$

② $\frac{d}{dx}\cos^{-1}u = -\frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$

③ $\frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2} \cdot \frac{du}{dx}$

④ $\frac{d}{dx}\cot^{-1}u = -\frac{1}{1+u^2} \cdot \frac{du}{dx}$

⑤ $\frac{d}{dx}\sec^{-1}u = \frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$

⑥ $\frac{d}{dx}\csc^{-1}u = -\frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$

These derivatives lead to the following integration formulas:

① $\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}u + C$

② $\int \frac{-du}{\sqrt{1-u^2}} = \cos^{-1}u + C$

③ $\int \frac{du}{1+u^2} = \tan^{-1}u + C$

④ $\int \frac{-du}{1+u^2} = \cot^{-1}u + C$

⑤ $\int \frac{du}{u\sqrt{u^2-1}} = \int \frac{d(-u)}{(-u)\sqrt{u^2-1}} = \sec^{-1}|u| + c = \cos^{-1}\left|\frac{1}{u}\right| + c$

⑥ $\int \frac{-du}{u\sqrt{u^2-1}} = \int \frac{-d(-u)}{(-u)\sqrt{u^2-1}} = \csc^{-1}|u| + c = \sin^{-1}\left|\frac{1}{u}\right| + c$

Ex.1: Find y' for $y = \sin^{-1}\sqrt{x}$

sol. $$\frac{dy}{dx} = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}\,\sqrt{1-x}}$$

Ex.2: If $\sin^{-1} y + \sec^{-1} 3x = 1$, find y'.

sol. $$\frac{1}{\sqrt{1-y^2}} \cdot \frac{dy}{dx} + \frac{1}{|3x|\sqrt{(3x)^2-1}}(3) = 0$$

$$\frac{1}{\sqrt{1-y^2}} \cdot \frac{dy}{dx} = \frac{-\cancel{3}}{\cancel{3}|x|\sqrt{9x^2-1}}$$

$$\Longrightarrow \frac{dy}{dx} = \frac{\frac{-1}{|x|\sqrt{9x^2-1}}}{\frac{1}{\sqrt{1-y^2}}} = \frac{-\sqrt{1-y^2}}{|x|\sqrt{9x^2-1}}$$

Ex.3: Evaluate $\int_0^1 \frac{dx}{1+x^2}$

sol. $$\int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^1 = \left[\tan^{-1} 1 - \tan^{-1} 0\right] = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

Ex.4: Evaluate $\int \frac{x^2\,dx}{\sqrt{1-x6}}$

sol. Let $u = x^3 \Rightarrow du = 3x^2\,dx \Rightarrow x^2\,dx = \frac{du}{3}$

$$\longrightarrow \frac{1}{3}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{3}\sin^{-1} u + c = \frac{1}{3}\sin^{-1}(x^3) + c$$

Ex.5: Evaluate $\int \frac{dx}{\sqrt{9-x^2}}$

sol. $$\sqrt{9-x^2} = \sqrt{9(1-\frac{x^2}{9})} = 3\sqrt{1-(\frac{x}{3})^2}$$

Let $u = \frac{x}{3} \Rightarrow du = \frac{1}{3}dx \Longrightarrow dx = 3\,du$

$$\longrightarrow \int \frac{3\,du}{3\sqrt{1-u^2}} = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + c = \sin^{-1}\left(\frac{x}{3}\right) + c$$

ercises 7.8 / P. 503 :

Evaluate :

(25) $\int_0^{\sqrt{2}/2} \frac{x\,dx}{\sqrt{1-x^4}}$

sol. Let $u = x^2 \Rightarrow du = 2x\,dx \Rightarrow x\,dx = \frac{du}{2}$

$U.L. = (\frac{\sqrt{2}}{2})^2 = \frac{2}{4} = \frac{1}{2}$

$L.L. = (0)^2 = 0$

$\longrightarrow \int_0^{1/2} \frac{du}{2\sqrt{1-u^2}} = \frac{1}{2}\int_0^{1/2} \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}[\sin^{-1}u]_0^{1/2}$

$= \frac{1}{2}[\sin^{-1}\frac{1}{2} - \sin^{-1}0] = \frac{1}{2}[\frac{\pi}{6} - 0] = \frac{1}{2}(\frac{\pi}{6}) = \frac{\pi}{12}$

(33) $\int_1^{\sqrt{3}} \frac{2\,dx}{(1+x^2)\tan^{-1}x}$

sol. Let $u = \tan^{-1}x \Rightarrow du = \frac{1}{1+x^2}dx$

$U.L. = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$

$L.L. = \tan^{-1} 1 = \frac{\pi}{4}$

$\longrightarrow 2\int_{\pi/4}^{\pi/3} \frac{du}{u} = 2[\ln u]_{\pi/4}^{\pi/3} = 2[\ln\frac{\pi}{3} - \ln\frac{\pi}{4}]$

(34) $\int_0^{\ln\sqrt{3}} \frac{e^x\,dx}{1+e^{2x}}$

sol. Let $u = e^x \Rightarrow du = e^x dx$

$\Rightarrow 1 + e^{2x} = 1 + (e^x)^2 = 1 + u^2$

$U.L. = e^{\ln\sqrt{3}} = \sqrt{3}$

$L.L. = e^0 = 1$

$\to \int_1^{\sqrt{3}} \frac{du}{1+u^2} = [\tan^{-1}u]_1^{\sqrt{3}} = [\tan^{-1}\sqrt{3} - \tan^{-1}1] = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$

(35) $\int_2^4 \frac{dx}{2x\sqrt{x-1}}$

<u>sol.</u> Let $u^2 = x \Rightarrow 2u\frac{du}{dx} = 1 \Rightarrow 2u\,du = dx$

$\Rightarrow \frac{dx}{2x} = \frac{2u\,du}{2x} = \frac{2u\,du}{2u^2} = \frac{du}{u}$

and $\sqrt{x-1} = \sqrt{u^2-1}$

$U.L. = \sqrt{4} = 2$ $\quad$ [As $u^2 = x \Rightarrow u = \sqrt{x}$]

$L.L. = \sqrt{2}$

$\rightarrow \int_{\sqrt{2}}^{2} \frac{du}{u\sqrt{u^2-1}} = \left[\sec^{-1}|u|\right]_{\sqrt{2}}^{2} = \sec^{-1}2 - \sec^{-1}\sqrt{2}$

$= \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$

CHAPTER EIGHT

TECHNIQUES OF INTEGRAT ON

Integration is not as straightforward as differentiation, there are no rules that absolutely guarantee obtaining an integral of a function.

In this chapter, we will study techniques for using basic integration formulas to obtain integrals. Also we will study techniques to obtain integrals of more complicated functions.

8.1 : Basic Integration Formulas:

In this section, we use techniques to make the integrands in the forms of basic integration formulas.

① Substitution:

Ex. Evaluate $\int \sqrt{1+x^2} \cdot x^5\, dx$.

Sol. $\int \sqrt{1+x^2} \cdot x^5\, dx = \int \sqrt{1+x^2} \cdot x^4 \cdot x\, dx$

Let $u = 1+x^2 \Rightarrow du = 2x\,dx \Rightarrow x\,dx = \frac{du}{2}$.

$x^2 = u-1 \Rightarrow x^4 = (x^2)^2 = (u-1)^2$

$\longrightarrow \int \sqrt{u} \cdot (u-1)^2 \cdot \frac{du}{2} = \frac{1}{2} \int u^{1/2} \cdot (u^2-2u+1) \cdot du$

$= \frac{1}{2} \int (u^{5/2} - 2u^{3/2} + u^{1/2})\, du$

$= \frac{1}{2} \left[\frac{2}{7} u^{7/2} - \frac{4}{5} u^{5/2} + \frac{2}{3} u^{3/2} \right] + c$

$= \frac{1}{7} (1+x^2)^{7/2} - \frac{2}{5} (1+x^2)^{5/2} + \frac{1}{3} (1+x^2)^{3/2} + c$

② Completing the square:

We want to write (ax^2+bx+c) in the form (au^2+k).

We have ax^2+bx+c :

1- Factor out a from first two terms :

$$\Rightarrow a\left(x^2+\frac{b}{a}x\right)+c$$

2- Add and subtract the square of half coefficient of x $\left[\text{or } \left(\frac{1}{2}\cdot\frac{b}{a}\right)^2 = \frac{b^2}{4a^2}\right]$:

$$\Rightarrow a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}\right)+c$$

3- Bring out the $\frac{-b^2}{4a^2}$:

$$\Rightarrow a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}\right)-\cancel{a}\left(\frac{b^2}{4a^{\cancel{2}}}\right)+c$$

$$= a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}\right)+\left(c-\frac{b^2}{4a}\right)$$

$$= a\left(x+\frac{b}{2a}\right)^2+\left(c-\frac{b^2}{4a}\right)$$

$$= au^2+k$$

Where: $u = x+\frac{b}{2a}$

$k = \left(c-\frac{b^2}{4a}\right)$

Ex.1 x^2+2x+3

$a=1$, $b=2$, $c=3$

$$\Rightarrow u = x+\frac{b}{2a} = x+\frac{2}{2(1)} = x+1$$

$$k = c-\frac{b^2}{4a} = 3-\frac{(2)^2}{4(1)} = 3-1 = 2$$

$$\Rightarrow x^2+2x+3 = au^2+k = (x+1)^2+2$$

Ex.2 $3x^2-4x+5$

$a=3\ ,\ b=-4\ ,\ c=5$

$u=x+\frac{b}{2a}=x+\frac{-4}{2(3)}=x-\frac{2}{3}$

$k=c-\frac{b^2}{4a}=5-\frac{(-4)^2}{4(3)}=5-\frac{4}{3}=\frac{11}{3}$

$\Rightarrow 3x^2-4x+5=3\left(x-\frac{2}{3}\right)^2+\frac{11}{3}$

Ex.3 Evaluate $\int\frac{dx}{x^2+2x+2}$

sol. $x^2+2x+2=a\left(x+\frac{b}{2a}\right)^2+\left(c-\frac{b^2}{4a}\right)$

$=(x+1)^2+1$

$\rightarrow \int\frac{dx}{(x+1)^2+1}$

Let $u=x+1 \Rightarrow du=dx$

$\rightarrow \int\frac{du}{u^2+1}=\tan^{-1}u+c=\tan^{-1}(x+1)+c$

---

③ Expanding a power and using trigonometric identity:

Ex. $\int(\sec x+\tan x)^2\,dx$

sol. $=\int(\sec^2x+2\sec x\tan x+\tan^2x)\,dx$

$=\int(\sec^2x+2\sec x\tan x+\underbrace{\sec^2x-1}_{})\,dx$

$=\int(2\sec^2x+2\sec x\tan x-1)\,dx$

$=2\tan x+2\sec x-x+c$

Note:
$\tan^2x+1=\sec^2x$
$\Rightarrow \tan^2x=\sec^2x-1$

④ Reducing an improper fraction:

Improper fraction is the fraction with degree of numerator equals or greater than the degree of denominator. The long division is used to integrate this fraction.

Ex. Evaluate $\int \frac{dx}{\sqrt[3]{x}-\sqrt{x}}$

sol. Let $x = u^6 \Rightarrow 6u^5\,du = dx$

$$\Rightarrow \sqrt[3]{x} = (u^6)^{1/3} = u^2$$

$$\Rightarrow \sqrt{x} = (u^6)^{1/2} = u^3$$

$$\longrightarrow \int \frac{6u^5\,du}{u^2-u^3} = 6\int \frac{u^5\,du}{u^2(1-u)} = 6\int \frac{u^3}{1-u} \quad \left[\begin{array}{l}\text{improper} \\ \text{fraction}\end{array}\right.$$

$$= 6\int \frac{-u^3\,du}{-(1-u)} = -6\int \frac{u^3\,du}{u-1} \quad \text{(Use long division)}$$

$$\longrightarrow -6\int\left(u^2+u+1+\frac{1}{u-1}\right)du$$

$$= -6\left[\frac{1}{3}u^3+\frac{1}{2}u^2+u+\ln|u-1|\right]+c$$

$$= -6\left[\frac{1}{3}\sqrt{x}+\frac{1}{2}\sqrt[3]{x}+\sqrt[6]{x}+\ln|\sqrt[6]{x}-1|\right]+c$$

$$\begin{array}{r|l} & u^2+u+1 \\ \hline u-1 & u^3 \\ & \mp u^3 \pm u^2 \\ \hline & u^2 \\ & \mp u^2 \pm u \\ \hline & u \\ & \mp u \pm 1 \\ \hline & 1 \end{array}$$

---

⑤ Separating a fraction:

Ex. Evaluate $\int \frac{x^2+3}{x}\,dx$

sol. $\int \frac{x^2+3}{x}\,dx = \int\left(\frac{x^2}{x}+\frac{3}{x}\right)dx = \int\left(x+\frac{3}{x}\right)dx$

$$= \frac{1}{2}x^2 + 3\ln|x| + c$$

⑥ The secant integral:

$$\int \sec u\,du = \int \sec u \,\frac{\sec u + \tan u}{\sec u + \tan u}\,du = \int \frac{\sec^2 u + \sec u \tan u}{\sec u + \tan u}\,du$$

$$\Rightarrow \boxed{\int \sec u\,du = \ln|\sec u + \tan u| + C}$$

Ex. $\int \sec 3x\,dx = \frac{1}{3}\int \sec 3x \cdot 3dx = \frac{1}{3}\ln|\sec 3x + \tan 3x| + C$

⑦ The cosecant integral:

$$\int \csc u\,du = \int \csc u \,\frac{\csc u + \cot u}{\csc u + \cot u}\,du = \int \frac{\csc^2 u + \csc u \cot u}{\csc u + \cot u}\,du$$

$$\Longrightarrow \boxed{\int \csc u\,du = -\ln|\csc u + \cot u| + C}$$

Ex. $\int \csc(\sin 5x) \cdot \cos 5x\,dx$

$$= \frac{1}{5}\int \csc(\sin 5x) . 5\cos 5x\,dx$$

$$= -\frac{1}{5}\ln|\csc(\sin 5x) + \cot(\sin 5x)| + C$$

⑧ Sequences of substitutions:

Ex. $\int \sqrt{1+\sin^2(x-1)} \cdot \sin(x-1) \cdot \cos(x-1) \cdot dx$

Sol. Let $u = \sin(x-1) \Rightarrow du = \cos(x-1)\,dx$

$$\rightarrow \int \sqrt{1+u^2} \cdot u \cdot du$$

Now let $v = 1+u^2 \Longrightarrow dv = 2u\,du \Rightarrow u\,du = \frac{dv}{2}$

$$\rightarrow \int \sqrt{v} \cdot \frac{dv}{2} = \frac{1}{2}\int v^{1/2} . dv = \frac{1}{2}\left(\frac{2}{3}\right) v^{3/2} + c$$

$$= \frac{1}{3} v^{3/2} + c = \frac{1}{3}(1+u^2)^{3/2} + c = \frac{1}{3}\left[1+\sin^2(x-1)\right]^{3/2} + c$$

8.2 : Integration by Parts :

If $u = f(x)$ and $v = g(x)$:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrate both sides with respect to dx :

$$\Rightarrow \quad d(uv) = u\,dv + v\,du$$

$$\Rightarrow \quad u\,dv = d(uv) - v\,du$$

Integrate both sides $\Rightarrow \int u\,dv = \int d(uv) - \int v\,du$

$$\Rightarrow \boxed{\int u\,dv = uv - \int v\,du}$$

the formula for integration by parts.

The object of this formula is to go from the given integral ($\int u\,dv$) to a new integral ($\int v\,du$) that is simpler.

(u) is chosen in which can be differentiated repeatedly to become zero, or chosen in which can be appear repeatedly after differentiation.

(dv) is chosen in which can be integrated repeatedly without difficulty.

In definite integral, the formula for integration by parts becomes :

$$\boxed{\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du}$$

Ex.1: Evaluate $\int x \cos x \, dx$.

Sol. $\int u\,dv = uv - \int v\,du$

Let $u = x \Rightarrow du = dx$

and $dv = \cos x \, dx \Rightarrow v = \int \cos x \, dx = \sin x$

$$\Rightarrow \int x \cos x \, dx = x \sin x - \int \sin x \, dx$$

$$= x \sin x + \cos x + c$$

Ex.2: Evaluate $\int \ln x \, dx$

Sol. Let $u = \ln x \Rightarrow du = \frac{dx}{x}$

$dv = dx \Rightarrow v = x$

$$\Rightarrow \int \ln x \, dx = x \cdot \ln x - \int x \cdot \frac{dx}{x}$$

$$= x \cdot \ln x - \int dx$$

$$= x \cdot \ln x - x + c$$

Ex.3: Evaluate $\int e^{\sqrt{x}} \, dx$

Sol. Let $x = w^2 \Rightarrow 2w\,dw = dx$ and $\sqrt{x} = w$

$$\rightarrow \int e^{w} \cdot 2w\,dw = 2\int e^{w} \cdot w\,dw = 2\int w \cdot e^{w} dw$$

Now use the integration by parts:

Let $u = w \Rightarrow du = dw$

$dv = e^{w} dw \Rightarrow v = e^{w}$

$$2\int u\,dv = 2\left[uv - \int v\,du\right]$$

$$2\int w \cdot e^{w} dw = 2\left[w \cdot e^{w} - \int e^{w} dw\right] = 2\left[w \cdot e^{w} - e^{w}\right] + c$$

$$= 2\left[\sqrt{x} \cdot e^{\sqrt{x}} - e^{\sqrt{x}}\right] + c = 2e^{\sqrt{x}}(\sqrt{x} - 1) + c$$

Repeated use:

Sometimes we have to use integration by parts more than once to obtain the answer.

Ex.4: Evaluate $\int x^2 e^x \, dx$.

Sol. Let $u = x^2 \Rightarrow du = 2x\,dx$

$dv = e^x dx \Rightarrow v = e^x$

$$\Rightarrow \int x^2 e^x dx = x^2 \cdot e^x - 2\int e^x \cdot x \cdot dx \quad \text{———} \circledast$$

Now $\int e^x \cdot x\,dx$: Use (by parts) again:

Let $u = x \Rightarrow du = dx$

$dv = e^x dx \Rightarrow v = e^x$

$$\Rightarrow \int e^x \cdot x\,dx = x e^x - \int e^x dx = x e^x - e^x + c_1$$

Return to main integral $\circledast$:

$$\Rightarrow \int x^2 e^x dx = x^2 \cdot e^x - 2(x e^x - e^x + c_1)$$

$$= x^2 e^x - 2x e^x + 2e^x + c \qquad [c = -2c_1]$$

Tabular Integration:

This is also integration by parts. This integration is used when the integration by parts required many repititions.

For last example: $\int x^2 e^x dx$:

Let $f(x) = x^2$ and $g(x) = e^x$

f(x) and its derivatives		g(x) and its integrals
x^2	$\oplus$	e^x
$2x$	$\ominus$	e^x
2	$\oplus$	e^x
0		e^x

$$\Rightarrow \int x^2 e^x dx = x^2 e^x - 2x e^x + 2e^x + c$$

(Same result)

Ex.2: Evaluate $\int x^3 \sin x \, dx$.

sol. Let $u = f(x) = x^3$

Let $v = g(x) = \sin x$

$$\Rightarrow \int x^3 \sin x = \int f(x) . g(x) . dx$$

Make a table :

f(x) and its derivative		g(x) and its integrals
x^3	$\oplus$	$\sin x$
$3x^2$	$\ominus$	$-\cos x$
$6x$	$\oplus$	$-\sin x$
6	$\ominus$	$\cos x$
0		$\sin x$

$$\Rightarrow \int x^3 \sin x = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C$$

Exercises 8.2 / P.534:

⑦ Evaluate $\int \tan^{-1} x \, dx$.

sol. Let $u = \tan^{-1} x \Rightarrow du = \frac{1}{1+x^2} dx$

$dv = dx \quad \Rightarrow v = x$

$$\Rightarrow \int \tan^{-1} x \, dx = x \tan^{-1} x - \int x \left(\frac{1}{1+x^2}\right) dx$$

$$= x \tan^{-1} x - \frac{1}{2} \ln |1+x^2| + C$$

㉑ Evaluate $\int e^x \sin x \, dx$.

Sol. Let $u = e^x \Rightarrow du = e^x dx$

$dv = \sin x \, dx \Rightarrow v = -\cos x$

$$\Rightarrow \int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx \quad \text{———} \circledast$$

Now: $\int e^x \cos x \, dx$

Let $u = e^x \Rightarrow du = e^x dx$

$dv = \cos x \, dx \Rightarrow v = \sin x$

$$\Rightarrow \int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$$

Return to main integral $\circledast$:

$$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$$

$$2 \int e^x \sin x \, dx = e^x (\sin x - \cos x)$$

$$\Rightarrow \int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + C$$

8.3 : Trigonometric Integrals :

① Power Products of Sine and Cosine :

We have the form : $\int \sin^m x \cdot \cos^n x \, dx$.

m and n are positive.

To evaluate this integral, there are three cases :

CASE 1 : m is odd :

In this case, save one sine factor and use $\sin^2 x = 1 - \cos^2$ to express the remaining factors in terms of cosine. Then susbstitute $u = \cos x$.

Ex. Evaluate $\int \sin^5 x \cdot \cos^2 x \, dx$.

sol. $= \int \sin x \cdot \sin^4 x \cdot \cos^2 x \, dx$

$= \int \sin x \cdot (1-\cos^2 x)^2 \cdot \cos^2 x \, dx$

Let $u = \cos x \Rightarrow du = -\sin x \, dx \Rightarrow \sin x \, dx = -du$

$\longrightarrow -\int (1-u^2)^2 \cdot u^2 \cdot du$

$= -\int (1-2u^2+u^4) \cdot u^2 \, du = -\int (u^2 - 2u^4 + u^6) \, du$

$= -\left[\frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7\right] + C = -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 + C$

CASE 2: n is odd :

In this case, save one cosine factor and use ($\cos^2 x = 1 - \sin^2 x$) to express the remaining factors in terms of sine. Then substitute $u = \sin x$.

Ex. Evaluate $\int \cos^5 x \cdot dx$

sol. $= \int \cos x \cdot \cos^4 x \, dx$

$= \int \cos x \, (1 - \sin^2 x)^2 \, dx$

Let $u = \sin x \Rightarrow du = \cos x \, dx$

$\rightarrow \int (1-u^2)^2 \cdot du = \int (1 - 2u^2 + u^4) \, du$

$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + c$

$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$

CASE 3: Both m and n are even:

In this case, use $\sin^2 x = \frac{1-\cos 2x}{2}$ and $\cos^2 x = \frac{1+\cos 2x}{2}$

Ex. Evaluate $\int \sin^2 x \cos^4 x \, dx$

Sol.
$$= \int \left(\frac{1-\cos 2x}{2}\right) \cdot \left(\frac{1+\cos 2x}{2}\right)^2 \cdot dx$$
$$= \frac{1}{8}\int (1-\cos 2x) \cdot (1+2\cos 2x+\cos^2 2x) \cdot dx$$
$$= \frac{1}{8}\int (1+\cos 2x-\cos^2 2x-\cos^3 2x)\, dx \quad \text{——} \circledast$$

Now: $\int \cos^2 2x \, dx = \frac{1}{2}\int (1+\cos 4x)\, dx = \frac{1}{2}(x+\frac{1}{4}\sin 4x)$

And: $\int \cos^3 2x \, dx = \int (1-\sin^2 2x)\cos 2x \, dx$

Let $u = \sin 2x \Rightarrow du = 2\cos 2x \, dx \Rightarrow \cos 2x \, dx = \frac{du}{2}$

$$\rightarrow \frac{1}{2}\int (1-u^2)\, du = \frac{1}{2}\left[u - \frac{1}{3}u^3\right]$$
$$= \frac{1}{2}\left[\sin 2x - \frac{1}{3}\sin^3 2x\right].$$

Return to main integral $\circledast$:

$$\Rightarrow \int \sin^2 x \cdot \cos^4 x \cdot dx = \frac{1}{8}\left[x + \frac{1}{2}\sin 2x - \frac{1}{2}(x+\frac{1}{4}\sin 4x) - \frac{1}{2}(\sin 2x - \frac{1}{3}\sin^3 2x)\right] + C$$

$$= \frac{1}{8}\left[x + \frac{1}{2}\sin 2x - \frac{1}{2}x - \frac{1}{8}\sin 4x - \frac{1}{2}\sin 2x + \frac{1}{6}\sin^3 2x\right] + C$$
$$= \frac{1}{16}\left(x - \frac{1}{4}\sin 4x + \frac{1}{3}\sin^3 2x\right) + C$$

Note: If both m and n are odd, either case 1 or case 2 can be used.

② Products of Sine and Cosine:

To evaluate the integrals : $\int \sin mx \cdot \cos nx \cdot dx$, $\int \sin mx \cdot \sin nx \cdot dx$, and $\int \cos mx \cdot \cos nx \cdot dx$

use the identities:

$$\sin mx \cdot \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x]$$
$$\sin mx \cdot \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]$$
$$\cos mx \cdot \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]$$

These identities come from the identities $[\cos(A\mp B)$ & $\sin(A\mp$

Ex. Evaluate $\int \sin 3x \cos 5x \, dx$.

Sol. $m=3$, $n=5$

$$\Rightarrow \int \sin 3x \cos 5x \, dx = \int \frac{1}{2}[\sin(3-5)x + \sin(3+5)x]dx$$
$$= \frac{1}{2}\int [\sin(-2x) + \sin(8x)]dx$$

$\sin(-x) = -\sin x \Rightarrow$
$$= \frac{1}{2}\int [\sin 8x - \sin 2x]\,dx$$
$$= \frac{1}{2}\left[\frac{-1}{8}\cos 8x + \frac{1}{2}\cos 2x\right] + c$$
$$= \frac{-1}{16}\cos 8x + \frac{1}{4}\cos 2x + C$$
$$= \frac{1}{4}\cos 2x - \frac{1}{16}\cos 8x + C$$

③ Powers of Secant and Tangent:

CASE 1: Odd power of secant:

In this case, use integration by parts, and the identity $(\tan^2 x = \sec^2 x - 1)$.

Ex. Evaluate $\int \sec^3 x \, dx$.

sol. $\int \sec^3 x \, dx = \int \sec x . \sec^2 x \, dx$

Use *integration by parts*:

Let $u = \sec x \Rightarrow du = \sec x \tan x \, dx$

$dv = \sec^2 x \, dx \Rightarrow v = \tan x$

$$\Rightarrow \int \sec^3 x \, dx = \sec x \tan x - \int \tan x . \sec x . \tan x . dx$$

$$= \sec x \tan x - \int \tan^2 x . \sec x . dx$$

$$= \sec x \tan x - \int (\sec^2 x - 1) . \sec x . dx$$

$$= \sec x \tan x - \int (\sec^3 x - \sec x) dx$$

$$\Rightarrow \int \sec^3 x \, dx = \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

$$2 \int \sec^3 x \, dx = \sec x \tan x + \ln |\sec x + \tan x|$$

$$\Rightarrow \int \sec^3 x \, dx = \frac{1}{2} [\sec x \tan x + \ln |\sec x + \tan x|] + c$$

CASE 2: Even power of secant:

In this case, save $\sec^2 x$ and use $(\sec^2 x = \tan^2 x + 1)$

Ex. Evaluate $\int \sec^4 x \, dx$.

sol. $\int \sec^4 x \, dx = \int \sec^2 x \cdot \sec^2 x \, dx$

$= \int (\tan^2 x + 1) \cdot \sec^2 x \, dx$

$= \int [\tan^2 x \cdot \sec^2 x + \sec^2 x] dx$

$= \frac{1}{3} \tan^3 x + \tan x + C$

CASE 3: Odd and even powers of tangent:

In this case, save $\tan^2 x$ and use $(\tan^2 x = \sec^2 x - 1)$.

Ex. Evaluate $\int \tan^5 x \, dx$

sol. $= \int \tan^2 x \cdot \tan^3 x \cdot dx$

$= \int (\sec^2 x - 1) \cdot \tan^3 x \cdot dx$

$= \int (\sec^2 x \cdot \tan^3 x - \tan^3 x) \, dx$ ——— ⊛

Now $\int \tan^3 x \, dx = \int (\tan^2 x \cdot \tan x) dx$

$= \int (\sec^2 x - 1) \cdot \tan x \cdot dx$

$= \int (\sec^2 x \cdot \tan x - \tan x) dx = \frac{1}{2} \tan^2 x - \ln|\sec x|$

Return to ⊛:

$\Rightarrow \int \tan^5 x \, dx = \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x + \ln|\sec x| + C$

④ Power Products of Secant and Tangent:

We have the form $\int \sec^m x \cdot \tan^n x \cdot dx$

m and n are positive.

To evaluate this integral, there are three cases:

CASE 1: m is even:

In this case, save $\sec^2 x$ and use $(\sec^2 x = \tan^2 x + 1)$.
Then substitute $u = \tan x$.

Ex. Evaluate $\int \sec^4 x \cdot \tan x \cdot dx$

sol.
$$= \int \sec^2 x \cdot \sec^2 x \cdot \tan x \cdot dx$$
$$= \int (\tan^2 x + 1) \cdot \sec^2 x \cdot \tan x \cdot dx$$

Let $u = \tan x \Rightarrow du = \sec^2 x \, dx$

$$\Longrightarrow \int (u^2+1) \cdot u\,du = \int (u^3 + u)\,du = \frac{1}{4}u^4 + \frac{1}{2}u^2 + C$$
$$= \frac{1}{4}\tan^4 x + \frac{1}{2}\tan^2 x + C$$

CASE 2: m is odd and n is odd:

In this case, save $(\sec x \tan x)$ and use $(\tan^2 x = \sec^2 x - 1)$ for remaining factor. Then substitute $u = \sec x$.

Ex. Evaluate $\int \sec^3 x \tan^3 x \, dx$.

Sol.:
$$= \int (\sec x \tan x) \cdot \sec^2 x \cdot \tan^2 x \cdot dx$$
$$= \int (\sec x \tan x) \cdot \sec^2 x \cdot (\sec^2 x - 1) \cdot dx$$

Let $u = \sec x \Rightarrow du = \sec x \tan x \, dx$

$$\Rightarrow \int u^2 \cdot (u^2 - 1) \cdot du = \int (u^4 - u^2] \cdot du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C$$
$$= \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C$$

CASE 3: m is odd and n is even:

In this case, use $(\tan^2 x = \sec^2 x - 1)$.

Ex. Evaluate $\int \sec x \cdot \tan^2 x \cdot dx$

Sol. $= \int \sec x \cdot (\sec^2 x - 1) \cdot dx = \int (\sec^3 x - \sec x)\, dx$

Now $\int \sec^3 x\, dx = \frac{1}{2}[\sec x \tan x + \ln|\sec x + \tan x|]$

* From example / P. 14 (lectures) *

$$\Rightarrow \int \sec x \cdot \tan^2 x\, dx = \frac{1}{2}[\sec x \tan x + \ln|\sec x + \tan x|] - \ln|\sec x + \tan x| + c$$

Exercises 8.3 / P. 541:

㉗ Evaluate $\int_{\pi/4}^{\pi/2} \csc^4 \theta\, d\theta$

Sol. $= \int_{\pi/4}^{\pi/2} \csc^2\theta \cdot \csc^2\theta \cdot d\theta$

$= \int_{\pi/4}^{\pi/2} (\cot^2\theta + 1) \cdot \csc^2\theta \cdot d\theta$

$= \int_{\pi/4}^{\pi/2} [\cot^2\theta \cdot \csc^2\theta + \csc^2\theta]\, d\theta$

$= \left[-\frac{1}{3}\cot^3\theta - \cot\theta\right]_{\pi/4}^{\pi/2}$

$= \left[-\frac{1}{3}\cot^3\frac{\pi}{2} - \cot\frac{\pi}{2}\right] - \left[-\frac{1}{3}\cot^3\frac{\pi}{4} - \cot\frac{\pi}{4}\right]$

$= [-0 - 0] - \left[-\frac{1}{3} - 1\right] = 0 - \left(-\frac{4}{3}\right) = \frac{4}{3}$

8.4: Trigonometric Substitutions:

Trigonometric substitutions enable us to replace the binomials (a^2+u^2), (a^2-u^2), and (u^2-a^2) by single squared terms to find many important integrals.

In this substitutions:

For a^2-u^2 use $u=a\sin\theta$, $\frac{-\pi}{2}\leqslant\theta\leqslant\frac{\pi}{2}$

For a^2+u^2 use $u=a\tan\theta$, $\frac{-\pi}{2}<\theta<\frac{\pi}{2}$

For u^2-a^2 use $u=a\sec\theta$, $0\leqslant\theta\leqslant\pi$, $\theta\neq\frac{\pi}{2}$

Ex.1: Evaluate $\int\frac{\sqrt{9-x^2}}{x^2}\,dx$

Sol. $9-x^2 \Longleftrightarrow a^2-u^2 \Longrightarrow a^2=9 \Longrightarrow a=3$

$u^2=x^2 \Longrightarrow u=x$

use $u=a\sin\theta$

$\Longrightarrow x=3\sin\theta \Longrightarrow dx=3\cos\theta\,d\theta$.

and $\sqrt{9-x^2}=\sqrt{9-9\sin^2\theta}=\sqrt{9(1-\sin^2\theta)}=3\cos\theta$

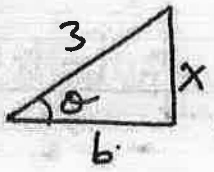
$$\Longrightarrow \int\frac{\sqrt{9-x^2}}{x^2}\,dx=\int\frac{3\cos\theta}{9\sin^2\theta}\,3\cos\theta\,d\theta=\int\frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$

$$=\int\cot^2\theta\,d\theta=\int(\csc^2\theta-1)\,d\theta=\boxed{-\cot\theta-\theta+C}$$

To replace θ by the variable x:

$x=3\sin\theta \Longrightarrow \theta=\sin^{-1}\left(\frac{x}{3}\right)$ \& $\sin\theta=\frac{x}{3}$

From the reference triangle:

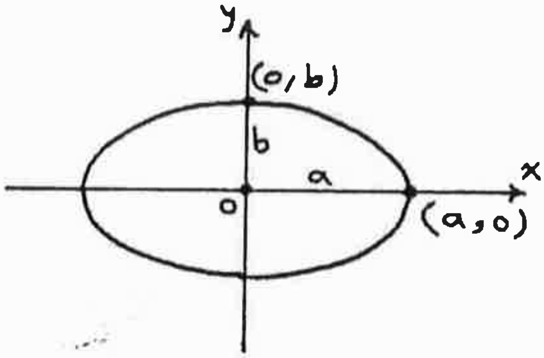


$b=\sqrt{(3)^2-x^2}=\sqrt{9-x^2} \Longrightarrow \cot\theta=\frac{\sqrt{9-x^2}}{x}$

$$\Longrightarrow \int\frac{\sqrt{9-x^2}}{x^2}\,dx=-\cot\theta-\theta+C=-\frac{\sqrt{9-x^2}}{x}-\sin^{-1}\left(\frac{x}{3}\right)+C$$

Ex. 2: Show that the area of the ellipse is $A = \pi ab$.

Sol.



The equation for the ellipse centered at the origin is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 .$$

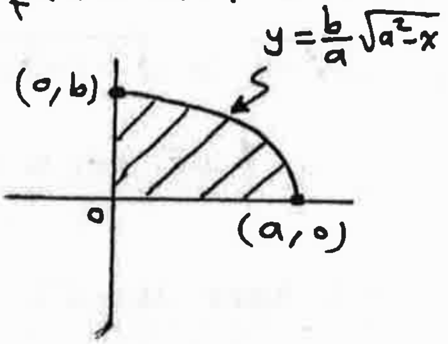
Write y as a function of x:

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2} \implies y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

$$\implies y = \mp \frac{b}{a}\sqrt{a^2 - x^2} .$$

Take the part of ellipse in the first quadrant:

Let the area under this part $= A_1$.



$$\implies A_1 = \int_0^a \frac{b}{a}\sqrt{a^2 - x^2} \quad \left[\text{from } A = \int_a^b f(x)dx\right]$$

Use $x = a\sin\theta \implies dx = a\cos\theta\, d\theta$

$$\implies \sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2(1 - \sin^2\theta)}$$

$$= \sqrt{a^2\cos^2\theta} = a\cos\theta$$

U.L.: $x = a \implies a = a\sin\theta \implies \sin\theta = \frac{a}{a} = 1$

$$\implies \theta = \sin^{-1} 1 = \frac{\pi}{2}$$

L.L.: $x = 0 \implies 0 = a\sin\theta \implies \theta = \sin^{-1}\frac{0}{a} = \sin^{-1} 0 = 0$

$$\implies A_1 = \frac{b}{a}\int_0^{\pi/2} a\cos\theta \, . \, a\cos\theta\, d\theta = ab\int_0^{\pi/2}\cos^2\theta\, d\theta$$

$$= ab\int_0^{\pi/2}\frac{1}{2}(1 + \cos 2\theta)\, d\theta = \frac{1}{2}ab\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2}$$

$$= \frac{1}{2}ab\left[\left(\frac{\pi}{2} + 0\right) - (0 + 0)\right] = \frac{\pi}{4}ab$$

$\implies$ Area of ellipse $= A = 4A_1 = 4\left(\frac{\pi}{4}ab\right) = \pi ab$

$$\implies A = \pi ab .$$

Exercises 8.4/ P. 547 :

㉑ Evaluate $\int \frac{dx}{\sqrt{x^2-2x}}$

Sol. $\int \frac{dx}{\sqrt{x^2-2x}} = \int \frac{dx}{\sqrt{x(x-2)}} = \int \frac{dx}{\sqrt{x}\,\sqrt{x-2}}$

Let $u^2 = x \Rightarrow 2u\,du = dx$

$\sqrt{x} = \sqrt{u^2} = u$

$\sqrt{x-2} = \sqrt{u^2-2}$

$\rightarrow \int \frac{2u\,du}{u\sqrt{u^2-2}} = 2\int \frac{du}{\sqrt{u^2-2}}$

$u^2-2 \Leftrightarrow u^2-a^2$ in which $a^2 = 2 \Rightarrow a = \sqrt{2}$

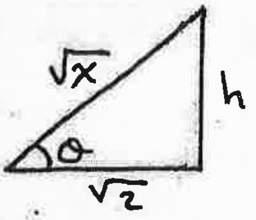
Use $u = a\sec\theta \Rightarrow u = \sqrt{2}\sec\theta \Rightarrow du = \sqrt{2}\sec\theta\tan\theta\,d\theta$

$\sqrt{u^2-2} = \sqrt{2\sec^2\theta - 2} = \sqrt{2(\sec^2\theta-1)} = \sqrt{2\tan^2\theta} = \sqrt{2}\tan\theta$

$\rightarrow 2\int \frac{\sqrt{2}\sec\theta\tan\theta\,d\theta}{\sqrt{2}\tan\theta} = 2\int \sec\theta\,d\theta = 2\ln|\sec\theta+\tan\theta|+c$

We have $u = \sqrt{2}\sec\theta \Rightarrow \sec\theta = \frac{u}{\sqrt{2}} = \frac{\sqrt{x}}{\sqrt{2}}$

From reference triangle:



$h = \sqrt{(\sqrt{x})^2 - (\sqrt{2})^2} = \sqrt{x-2}$

$\Rightarrow \tan\theta = \frac{\sqrt{x-2}}{\sqrt{2}}$

$\Rightarrow \int \frac{dx}{\sqrt{x^2-2x}} = 2\ln\left|\frac{\sqrt{x}}{\sqrt{2}} + \frac{\sqrt{x-2}}{\sqrt{2}}\right| + c$

$= \ln\left(\frac{\sqrt{x}}{\sqrt{2}} + \frac{\sqrt{x-2}}{\sqrt{2}}\right)^2 + c$

$= \ln\left(\frac{x}{2} + 2\,\frac{\sqrt{x}}{\sqrt{2}} * \frac{\sqrt{x-2}}{\sqrt{2}} + \frac{x-2}{2}\right) + c$

$= \ln\left(\frac{2x-2}{2} + \sqrt{x^2-2x}\right) + c$

$= \ln\left(\frac{2(x-1)}{2} + \sqrt{x^2-2x}\right) + c$

$= \ln\left(x-1+\sqrt{x^2-2x}\right) + c$

8.5: Rational Functions and Partial Fractions:

Sometimes we need to reverse the rational function to smaller partial fractions to find the integral.

If we have the rational function $\frac{f(x)}{g(x)}$. We can write the function $\frac{f(x)}{g(x)}$ as a sum of partial fractions if:

1- The degree of $f(x)$ is less than the degree of $g(x)$.
If it isn't use the long division.

2- We must know the factors of $g(x)$.

We will study four cases:

CASE 1: Distinct linear factors of $g(x)$:

$$\frac{f(x)}{g(x)} = \frac{A}{a_1x+b_1} + \frac{B}{a_2x+b_2} + \frac{C}{a_3x+b_3} + \cdots$$

Ex.1: Evaluate $\int \frac{5x-3}{x^2-2x-3}\,dx$

Sol. $\frac{5x-3}{x^2-2x-3} = \frac{5x-3}{(x+1)(x-3)}$

$$\frac{5x-3}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$$

Multiply both sides with $(x+1)(x-3)$:

$$\Rightarrow\ 5x-3 = A(x-3) + B(x+1)$$
$$5x-3 = Ax - 3A + Bx + B$$
$$5x-3 = (A+B)x - 3A + B$$

$$\Rightarrow\ \begin{array}{l} A+B = 5 \\ \pm 3A \mp B = \pm 3 \\ \hline 4A \quad = 8 \end{array}$$ subtract

$\Rightarrow A = 2$ & $B = 5-2 = 3$

$$\Rightarrow \int \frac{5x-3}{x^2-2x+3}\,dx = \int\left(\frac{2}{x+1}+\frac{3}{x-3}\right)dx$$

$$= 2\ln|x+1| + 3\ln|x-3| + k$$

Ex.2 : Evaluate $\displaystyle\int \frac{2x^3-4x^2-15x+5}{x^2-2x-8}\,dx$

Sol. First use long division [because deg. of $f(x) >$ deg. of $g(x)$]

$$\begin{array}{r|l} & 2x \\ \hline x^2-2x-8 & 2x^3-4x^2-15x+5 \\ & \mp 2x^3 \pm 4x^2 \pm 16x \quad \downarrow \\ \hline & \qquad\qquad\quad x+5 \end{array}$$

$$\Rightarrow \frac{2x^3-4x^2-15x+5}{x^2-2x-8} = 2x + \frac{x+5}{x^2-2x-8}$$

Now $\dfrac{x+5}{x^2-2x-8} = \dfrac{x+5}{(x+2)(x-4)}$

$$\Rightarrow \frac{x+5}{(x+2)(x-4)} = \frac{A}{x+2} + \frac{B}{x-4}$$

Multiply both sides with $(x+2)(x-4)$:

$$\Rightarrow x+5 = A(x-4) + B(x+2)$$

$$x+5 = Ax - 4A + Bx + 2B$$

$$x+5 = (A+B)x - 4A + 2B$$

$$\Rightarrow A+B=1 \quad [*-2] \Rightarrow -2A-2B=-2$$

$$-4A+2B=5 \xrightarrow{\qquad\qquad} -4A+2B=5$$

$$\underline{\qquad\qquad\qquad}\ \text{Add}$$

$$-6A=3$$

$$\Rightarrow A = -\frac{1}{2} \quad \& \quad B = 1-\left(-\frac{1}{2}\right) = \frac{3}{2}$$

$$\Rightarrow \int \frac{2x^3-4x^2-15x+5}{x^2-2x-8}\,dx = \int\left[2x + \frac{\frac{-1}{2}}{x+2} + \frac{\frac{3}{2}}{x-4}\right]dx$$

$$= x^2 - \frac{1}{2}\ln|x+2| + \frac{3}{2}\ln|x-4| + k$$

CASE 2 : Repeated linear factors of $g(x)$:

$$\frac{f(x)}{g(x)} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \frac{C}{(ax+b)^3} + \cdots$$

Ex.1: Evaluate $\int \frac{6x+7}{x^2+4x+4} dx$

Sol.

$$\frac{6x+7}{x^2+4x+4} = \frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

Multiply both sides with $(x+2)^2$:

$$\Rightarrow 6x+7 = A(x+2)+B$$

$$6x+7 = Ax+2A+B$$

$$\Rightarrow A=6 \quad \text{and} \quad 2A+B=7 \Rightarrow B=7-2A=7-2(6)=-5$$

$$\Rightarrow \int \frac{6x+7}{x^2+4x+4} dx = \int \left[\frac{6}{x+2} - \frac{5}{(x+2)^2} \right] dx = 6 \ln|x+2| + \frac{5}{x+2} + k$$

Ex.2 : $\int \frac{5x^2+20x+6}{x^3+2x^2+x} dx$

Sol. $\frac{5x^2+20x+6}{x^3+2x^2+x} = \frac{5x^2+20x+6}{x(x^2+2x+1)} = \frac{5x^2+20x+6}{x(x+1)^2}$

$$\Rightarrow \frac{5x^2+20x+6}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$ [Both CASE 1 & CASE 2]

Multiply both sides with $x(x+1)^2$:

$$\Rightarrow 5x^2+20x+6 = A(x+1)^2 + Bx(x+1) + Cx$$

$$5x^2+20x+6 = A(x^2+2x+1) + Bx^2 + Bx + Cx$$

$$5x^2+20x+6 = Ax^2+2Ax+A+Bx^2+Bx+Cx$$

$$5x^2+20x+6 = (A+B)x^2 + (2A+B+C)x + A$$

$$\Rightarrow A=6 \quad , \quad A+B=5 \Rightarrow B=-1 \quad , \quad 2A+B+C=20 \Rightarrow C=9$$

$$\Rightarrow \int \frac{5x^2+20x+6}{x^3+2x^2+x} dx = \int \left[\frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2} \right] dx = 6\ln|x| - \ln|x+1| - \frac{9}{x+1} + k$$

CASE 3: Distinct irreducible quadratic factors of $g(x)$:

$$\frac{f(x)}{g(x)} = \frac{Ax+B}{a_1x^2+b_1x+C_1} + \frac{Cx+D}{a_2x^2+b_2x+C_2} + \frac{Ex+F}{a_3x^2+b_3x+C_3} + \cdots$$

Ex. Evaluate $\int \frac{2x^2-5x+2}{x^3+x}\,dx$

Sol.: $\frac{2x^2-5x+2}{x^3+x} = \frac{2x^2-5x+2}{x(x^2+1)}$

$$\frac{2x^2-5x+2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$ [Both CASE 1 & CASE 3]

Multiply both sides with $x(x^2+1)$:

$$\Rightarrow 2x^2-5x+2 = A(x^2+1) + (Bx+C)(x)$$

$$2x^2-5x+2 = Ax^2+A+Bx^2+Cx$$

$$2x^2-5x+2 = (A+B)x^2+Cx+A$$

$$\Rightarrow A=2 \quad , \quad C=-5 \quad , \quad A+B=2 \Rightarrow B=2-A=2-2=0$$

$$\Rightarrow \int \frac{2x^2-5x+2}{x^3+x}\,dx = \int \left[\frac{2}{x} - \frac{5}{x^2+1}\right]dx = 2\ln|x| - 5\tan^{-1}x + k$$

CASE 4: Repeated irreducible quadratic factors of $g(x)$:

$$\frac{f(x)}{g(x)} = \frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{(ax^2+bx+c)^2} + \frac{Ex+F}{(ax^2+bx+c)^3} + \cdots$$

Ex. Evaluate $\displaystyle\int \frac{1-x+2x^2-x^3}{x^5+2x^3+x}\,dx$

Sol. $\displaystyle\frac{1-x+2x^2-x^3}{x^5+2x^3+x} = \frac{1-x+2x^2-x^3}{x(x^4+2x^2+1)} = \frac{1-x+2x^2-x^3}{x(x^2+1)^2}$

$$\Longrightarrow \frac{1-x+2x^2-x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2} \qquad [\text{CASE 1} \ \& \ \text{CASE 4}]$$

Multiply both sides with $x(x^2+1)^2$:

$$\Rightarrow 1-x+2x^2-x^3 = A(x^2+1)^2 + (Bx+C)(x)(x^2+1) + (Dx+E)(x)$$

$$1-x+2x^2-x^3 = A(x^4+2x^2+1) + (Bx^2+Cx)(x^2+1) + Dx^2+Ex$$

$$1-x+2x^2-x^3 = Ax^4+2Ax^2+A+Bx^4+Bx^2+Cx^3+Cx+Dx^2+Ex$$

$$1-x+2x^2-x^3 = (A+B)x^4+Cx^3+(2A+B+D)x^2+(C+E)x+A$$

$\Longrightarrow$ $A = 1$

$A+B=0 \Longrightarrow B=-1$

$C=-1$

$2A+B+D=2 \Longrightarrow D=1$

$C+E=-1 \Longrightarrow E=0$

$$\Rightarrow \int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx = \int \left[\frac{1}{x} + \frac{(-x-1)}{x^2+1} + \frac{x}{(x^2+1)^2}\right] dx$$

$$= \int \left[\frac{1}{x} - \frac{x}{x^2+1} - \frac{1}{x^2+1} + \frac{x}{(x^2+1)^2}\right] dx$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+1| - \tan^{-1}x - \frac{1}{2(x^2+1)} + k$$

The Substitution $z = \tan\left(\frac{x}{2}\right)$:

This substitution is used in rational functions that contain $(\sin x)$ and $(\cos x)$.

In this substitution, the values of $(\sin x)$ and $(\cos x)$ are replaced by the variable z.

The substitutions are:

$$\sin x = \frac{2z}{z^2+1}, \quad \cos x = \frac{1-z^2}{z^2+1}, \quad \text{and} \quad dx = \frac{2\,dz}{z^2+1}$$

Ex. Evaluate $\int \frac{dx}{1-\sin x}$

Sol. $$\int \frac{dx}{1-\sin x} = \int \frac{\frac{2dz}{z^2+1}}{1-\frac{2z}{z^2+1}} = \int \frac{\frac{2dz}{z^2+1}}{\frac{z^2+1-2z}{z^2+1}} = 2\int \frac{dz}{z^2-2z+1}$$

$$= 2\int \frac{dz}{(z-1)^2} = -\frac{2}{z-1} + c$$

$$= -\frac{2}{\tan\left(\frac{x}{2}\right)-1} + c$$

8.6: Using Integral Tables:

In this method we compare the integral with the tables of integral formulas.

For example: $\int \frac{dx}{x(3x+4)} = \frac{1}{4} \ln\left|\frac{x}{3x+4}\right| + c$

This solution comes from the following integral formula in the table:

$$\int \frac{dx}{x(ax+b)} = \frac{1}{b} \ln\left|\frac{x}{ax+b}\right| + c$$

where $a = 3$ and $b = 4$.

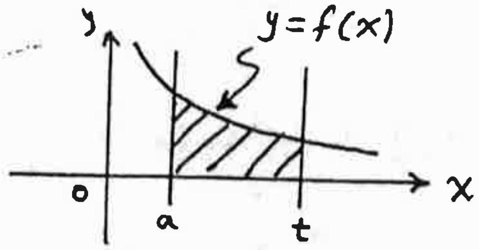
8.7: Improper Integrals:

The integral is called improper integral if :

1- The interval of integral is infinite.

2- The function (integrand) has discontinuity.

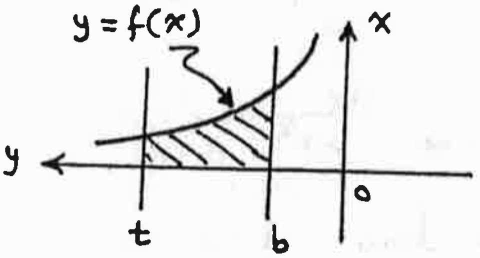
Type 1 : Infinite Intervals:



Definitions:

① If $\int_a^t f(x)\,dx$ exists then:

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx .$$



② If $\int_t^b f(x)\,dx$ exists then :

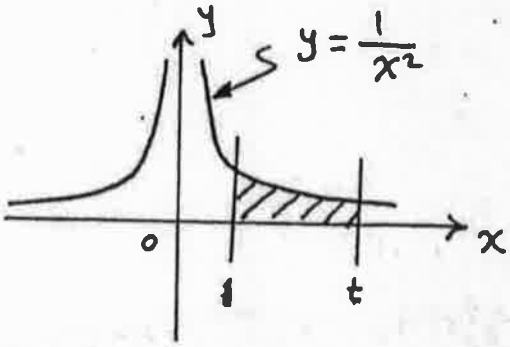
$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx .$$

③ The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called convergent if the limits are exist , and called divergent if the limits are not exist .

④ If $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent then:

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx .$$

Examples:



Ex.1: Evaluate $\int_1^\infty \frac{dx}{x^2}$.

Sol. $\int_1^t \frac{dx}{x^2} = -\left[\frac{1}{x}\right]_1^t = -\left[\frac{1}{t} - 1\right] = 1 - \frac{1}{t}$

$$\lim_{t\to\infty} \int_1^t \frac{dx}{x^2} = \lim_{t\to\infty} \left(1 - \frac{1}{t}\right) = 1 - \frac{1}{\infty} = 1 - 0 = 1$$

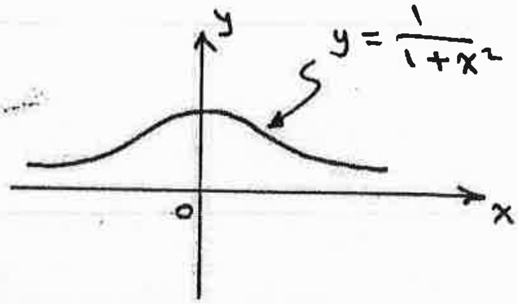
$$\Rightarrow \int_1^\infty \frac{dx}{x^2} = \lim_{t\to\infty} \int_1^t \frac{dx}{x^2} = 1$$

Ex.2: Evaluate $\int_1^{\infty} \frac{dx}{x}$.

Sol.: $\lim\limits_{t\to\infty} \int_1^t \frac{dx}{x} = \lim\limits_{t\to\infty} \left[\ln|x|\right]_1^t = \lim\limits_{t\to\infty} \left[\ln|t| - \ln 1\right]$.

$$= \lim_{t\to\infty} \ln|t| = \ln \infty = \infty$$

$\Rightarrow \int_1^{\infty} \frac{dx}{x}$ is divergent.



Ex.3: Evaluate $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

Sol.: $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^{\infty} f(x)\,dx$

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{t\to-\infty} \int_t^0 \frac{dx}{1+x^2} = \lim_{t\to-\infty} \left[\tan^{-1} x\right]_t^0$$

$$= \lim_{t\to-\infty} \left[\tan^{-1} 0 - \tan^{-1} t\right] = \lim_{t\to-\infty} \left[-\tan^{-1} t\right] = -\left(\frac{-\pi}{2}\right) = \frac{\pi}{2}$$

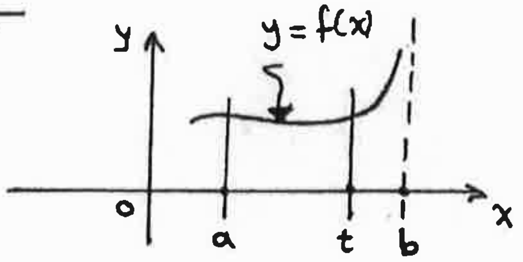
$$\int_0^{\infty} \frac{dx}{1+x^2} = \lim_{t\to\infty} \int_0^t \frac{dx}{1+x^2} = \lim_{t\to\infty} \left[\tan^{-1} x\right]_0^t = \lim_{t\to\infty} \left[\tan^{-1} t - \tan^{-1} 0\right]$$

$$= \lim_{t\to\infty} \left[\tan^{-1} t\right] = \frac{\pi}{2}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

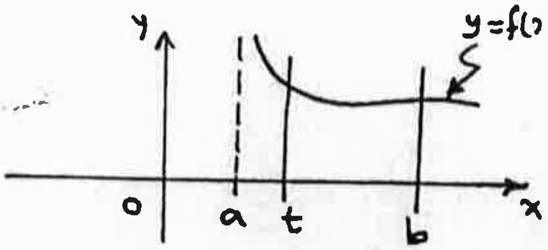
Type 2 : Discontinuous Integrands :

Definitions :



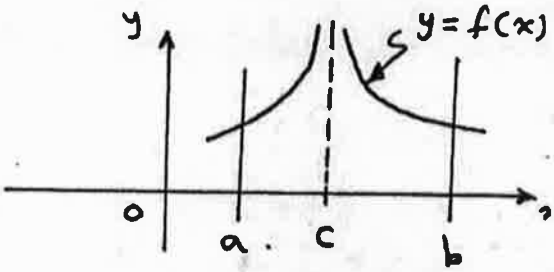
① For f(x) with [a, b) :

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$



② For f(x) with (a, b] :

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$



③ If f(x) discontinuous at c :

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Ex.1 : Evaluate $\int_2^5 \frac{dx}{\sqrt{x-2}}$

Sol. f(x) is discontinuous at $x = 2 \Rightarrow \int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,d$

$$\Rightarrow \int_2^5 \frac{dx}{\sqrt{x-2}} = \lim_{t \to 2^+} \int_t^5 \frac{dx}{\sqrt{x-2}} = \lim_{t \to 2^+} \left[2\sqrt{x-2}\,\right]_t^5$$

$$= \lim_{t \to 2^+} 2\left[\sqrt{3} - \sqrt{t-2}\,\right] = 2\left[\sqrt{3} - 0\right] = 2\sqrt{3}$$

Ex.2 : Evaluate $\int_0^3 \frac{dx}{x-1}$

Sol. : f(x) is discontinuous at $x = 1 \Rightarrow \int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$

Now : $\int_0^1 \frac{dx}{x-1} = \lim_{t \to 1^-} \int_0^t \frac{dx}{x-1} = \lim_{t \to 1^-} \left[\ln|x-1|\right]_0^t$

$$= \lim_{t \to 1^-} \left[\ln|t-1| - \ln|-1|\right] = \ln 0 - \ln 1 = \ln 0 = -\infty$$

$\Rightarrow \int_0^1 \frac{dx}{x-1}$ is divergent $\Rightarrow$ We do not need to evaluate $\int_1^3 \frac{dx}{x-1}$

$\Rightarrow \int_0^3 \frac{dx}{x-1}$ is divergent.

- بسم الله الرحمن الرحيم -

MATRICES , DETERMINANTS AND CRAMER'S RULE

Matrices : Matrices have many applications in science and engineering.

A rectangular array of numbers like $A = \begin{bmatrix} 4 & 2 & 1 \\ -2 & 0 & 3 \end{bmatrix}$ is called a matrix.

Matrix A has 2 rows and 3 columns, then A is called 2 by 3 matrix.

Each number is called element and denoted by a_{ij} where i : no. of row and j: no. of column.

For example $a_{11} = 4$, $a_{23} = 3$, $a_{13} = 1$, - - -

If the number of rows equals to the number of columns then the matrix is called square matrix. (matrix of order n in which n : no. of rows or columns

Equal matrices have identical corresponding elements.

For example $\begin{bmatrix} 2 & 4 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 1 & -2 \end{bmatrix}$

Ex. Find x , y and z :

$$\begin{bmatrix} x \\ x+2 \\ 2y-3 \end{bmatrix} = \begin{bmatrix} 4 \\ y \\ z \end{bmatrix}$$

sol.

$$x = 4$$

$$x+2 = y \Rightarrow 4+2 = y \Rightarrow y = 6$$

$$2y-3 = z \Rightarrow 2(6)-3 = z \Rightarrow z = 9$$

Diagonal Matrix: Is a square matrix where all the elements are zeroes except the principal (or main) diagonal.

For example: $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

Identity Matrix (I): Is a diagonal matrix with the elements of principal diagonal are ones.

For example: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ 2x2 identity matrix.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 3x3 identity matrix.

Addition and Subtraction of matrices:

Ex.1 (Addition): $\begin{bmatrix} 8 & 3 & 4 \\ 0 & -1 & 9 \end{bmatrix} + \begin{bmatrix} 5 & -2 & 1 \\ 6 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 13 & 1 & 5 \\ 6 & 2 & 14 \end{bmatrix}$

Ex.2 (Subtraction): $\begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ -3 & 4 \end{bmatrix}$

Multiplication a matrix by a constant:

Ex. $A = \begin{bmatrix} 3 & 1 \\ 7 & -1 \\ 2 & 8 \end{bmatrix}$

$\Rightarrow 3A = \begin{bmatrix} 9 & 3 \\ 21 & -3 \\ 6 & 24 \end{bmatrix}$

Multiplication of two matrices:

We can only multiply two matrices if the number of columns in first matrix equals to the number of rows in second matrix.

The resulting matrix has number of rows same as first matrix and number of columns same as second matrix.

For example: [2x3 matrix] * [3 x 4 matrix] = [2x4 matrix

How to multiply two matrices?

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} * \begin{bmatrix} u & v \\ w & x \\ y & z \end{bmatrix} = \begin{bmatrix} \overset{a_{11}}{au+bw+cy} & \overset{a_{12}}{av+bx+cz} \\ \underset{a_{21}}{du+ew+fy} & \underset{a_{22}}{dv+ex+fz} \end{bmatrix}$$

Ex.1:

$$\begin{bmatrix} 0 & -1 & 2 \\ 4 & 11 & 2 \end{bmatrix} * \begin{bmatrix} 3 & -1 \\ 1 & 2 \\ 6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0(3)-1(1)+2(6) & 0(-1)-1(2)+2(1) \\ 4(3)+11(1)+2(6) & 4(-1)+11(2)+2(1) \end{bmatrix} = \begin{bmatrix} 11 & 0 \\ 35 & 20 \end{bmatrix}$$

Notes: ① If A is a matrix then $A * I = A$.

② If A and B are matrices then $A * B$ may be not equals to $B * A$.

Ex.2: If possible, find AB and BA.

$$A = \begin{bmatrix} -2 & 1 & 7 \\ 3 & -1 & 0 \\ 0 & 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -1 & 5 \end{bmatrix}$$

Sol. AB not possible

$$BA = \begin{bmatrix} 4(-2)-1(3)+5(0) & 4(1)-1(-1)+5(2) & 4(7)-1(0)+5(-1) \end{bmatrix}$$

$$\Rightarrow BA = \begin{bmatrix} -11 & 15 & 23 \end{bmatrix}$$

Ex.3 Find :

$$\begin{bmatrix} \cos 60 & -\sin 60 & 0 \\ \sin 60 & \cos 60 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$$

Sol.

$$= \begin{bmatrix} \frac{1}{2}(2) - \frac{\sqrt{3}}{2}(4) + 0(0) \\ \frac{\sqrt{3}}{2}(2) + \frac{1}{2}(4) + 0(0) \\ 0(2) + 0(4) + 1(0) \end{bmatrix} = \begin{bmatrix} 1-2\sqrt{3} \\ \sqrt{3}+2 \\ 0 \end{bmatrix} \simeq \begin{bmatrix} -2.464 \\ 3.732 \\ 0 \end{bmatrix}$$

Determinants :

With each square matrix A we associate a number called determinant of A, denoted by (det. A) or (|A|), calculated from the elements of the matrix A.

How to calculate the determinant ?

First we define the minor and cofactor :

Minor : The minor of the element a_{ij} is the determinant that remains after we delete the row and column containing a_{ij}.

For example :

$$\begin{vmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ a_{21} & a_{22} & \cancel{a_{23}} \\ a_{31} & a_{32} & \cancel{a_{33}} \end{vmatrix}$$

the minor of a_{13} is $\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Cofactor : The cofactor of the element a_{ij} is $(-1)^{i+j}$ times the minor of a_{ij}. It's symbol is A_{ij}.

For the last determinant :

$$A_{22} = (-1)^{2+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

The signs for cofactors are:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Calculating the determinants:

① For 1 X 1 determinant:

$$|a| = a$$

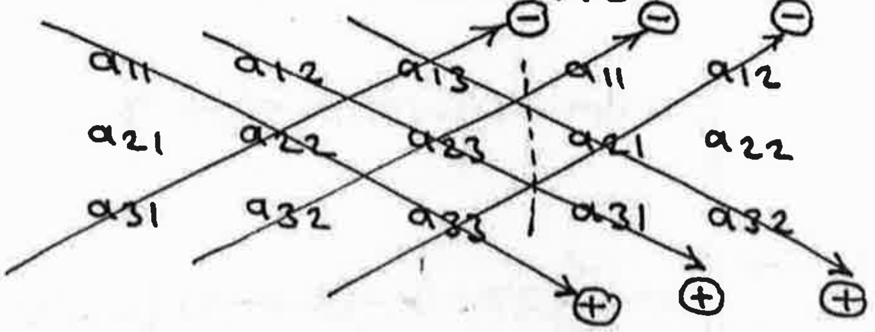
Ex. $|-4| = -4$

② For 2 X 2 determinant:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

Ex. $\begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix} = 4(3) - 2(1) = 10$

③ For 3 X 3 determinant:

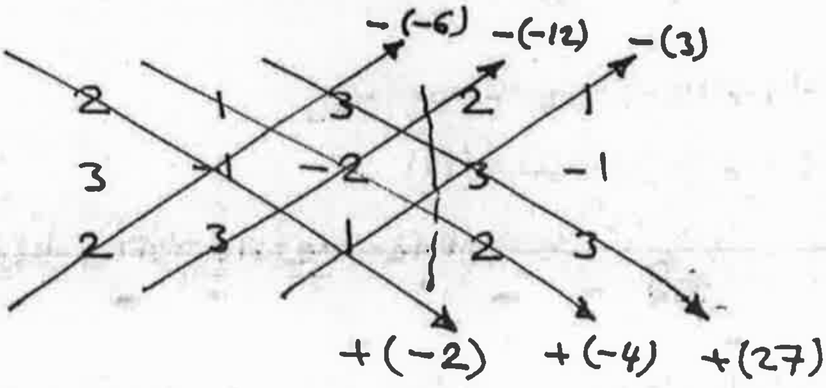


Copy the 1^{st} and 2^{nd} column.

Ex. If:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & -1 & -2 \\ 2 & 3 & 1 \end{bmatrix}$$ find det. A.

Sol.



$\Rightarrow$ det. A $= -2 - 4 + 27 + 6 + 12 - 3 = 36$

④ For determinants of any size:

This method called (expansion by minors):

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \quad \text{(from 1st row)}$$

$$\text{or} \quad = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} \quad \text{(from 2nd column)}$$

or from any row or column.

* To simplify the calculation, take row or column with greater number of zeroes.

Ex. Find the determinant for the matrix:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & -1 & -2 \\ 2 & 3 & 1 \end{bmatrix}$$

Sol. Take 1st row:

$$A_{11} = (-1)^{1+1} \begin{vmatrix} -1 & -2 \\ 3 & 1 \end{vmatrix} = (-1 * 1) - (3 * -2) = 5$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} = -\left[(3 * 1) - (2 * -2)\right] = -7$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix} = (3 * 3) - (2 * -1) = 11$$

$$\Rightarrow \text{det. } A = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$= 2(5) + 1(-7) + 3(11)$$

$$= 36$$

(same result with last example)

Transpose of a matrix (A^T):

It is the matrix obtained from interchanging the rows by columns.

Ex.

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -1 & 3 \\ 5 & 7 & 8 \end{bmatrix} \Longrightarrow A^T = \begin{bmatrix} 3 & 2 & 5 \\ 4 & -1 & 7 \\ 1 & 3 & 8 \end{bmatrix}.$$

The inverse of a matrix (A^{-1}):

The inverse matrix A^{-1} for the matrix A is the matrix in which $A * A^{-1} = I$.

We will study two methods to find A^{-1}.

Method 1 : By using the determinant of a matrix (for 2×2 matrices only):

Ex. Find A^{-1} for $A = \begin{bmatrix} 2 & -3 \\ 4 & -7 \end{bmatrix}$

sol.: ① Interchange main diagonal elements :

$$\Rightarrow \begin{bmatrix} -7 & -3 \\ 4 & 2 \end{bmatrix}$$

② Change signs of the other two elements:

$$\Rightarrow \begin{bmatrix} -7 & 3 \\ -4 & 2 \end{bmatrix}$$

③ Find det. A : $\Longrightarrow (2 * -7) - (4 * -3) = -2$

④ Multiply result from step ② by $\frac{1}{\text{det.} A}$ to get the inverse of the matrix :

$$\Rightarrow A^{-1} = \frac{1}{\text{det.} A} \begin{bmatrix} -7 & 3 \\ -4 & 2 \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} -7 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 3.5 & -1.5 \\ 2 & -1 \end{bmatrix}$$

For checking: $\begin{bmatrix} 2 & -3 \\ 4 & -7 \end{bmatrix} \begin{bmatrix} 3.5 & -1.5 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ o.k.

Method 2 : By using adjoint matrix method (For matrix of any size):

$$\boxed{A^{-1} = \frac{adj.\,A}{det.\,A}}$$ where adj. A : adjoint of A.

The adjoint matrix is found by replacing each element in the matrix with its cofactor, and then finding the transpose of the resulting matrix.

Ex.1: Find A^{-1} for $A = \begin{bmatrix} 5 & 6 & 1 \\ 0 & 3 & -3 \\ 4 & -7 & 2 \end{bmatrix}$

Sol.: ① Replace elements with cofactor:

$A_{11} = (3*2) - (-7*-3) = -15$

$A_{12} = 12$, $A_{13} = -12$, $A_{21} = 19$, $A_{22} = 6$, $A_{23} = -59$,

$A_{31} = -21$, $A_{32} = -15$, $A_{33} = 15$.

$$\Rightarrow \begin{bmatrix} +(-15) & -(12) & +(-12) \\ -(19) & +(6) & -(-59) \\ +(-21) & -(-15) & +(15) \end{bmatrix} = \begin{bmatrix} -15 & -12 & -12 \\ -19 & 6 & 59 \\ -21 & 15 & 15 \end{bmatrix}$$

② Transpose the matrix obtained from step ① to find adj. A :

$$\Rightarrow adj.\,A = \begin{bmatrix} -15 & -19 & -21 \\ -12 & 6 & 15 \\ -12 & 59 & 15 \end{bmatrix}$$

③ Find det. A :

$$\Rightarrow det.A = \begin{vmatrix} 5 & 6 & 1 \\ 0 & 3 & -3 \\ 4 & -7 & 2 \end{vmatrix} = 5(-15) + 4(-21) = -159$$

④ Find A^{-1} :

$$\Rightarrow A^{-1} = \frac{adj.\,A}{det.\,A} = \frac{1}{-159} \begin{bmatrix} -15 & -19 & -21 \\ -12 & 6 & 15 \\ -12 & 59 & 15 \end{bmatrix} = \begin{bmatrix} 0.094 & 0.119 & 0.132 \\ 0.075 & -0.038 & -0.094 \\ 0.075 & -0.371 & -0.094 \end{bmatrix}$$

Ex.2: Find A^{-1} for $A = \begin{bmatrix} -2 & 6 & 1 \\ 0 & 3 & -3 \\ 4 & -7 & 3 \end{bmatrix}$

Ans. $A^{-1} = \begin{bmatrix} 1/5 & 5/12 & 7/20 \\ 1/5 & 1/6 & 1/10 \\ 1/5 & -1/6 & 1/10 \end{bmatrix}$

Cramer's Rule:

Cramer's Rule is used to solve the simultaneous linear equations.

If we have the equations:

n = no. of rows.
= no. of columns

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1$$
$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n = b_n$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}$$ matrix of coefficients.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$ matrix of variables.

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$ matrix of constants.

$$\Longrightarrow \{A\}\{x\} = \{b\}$$

$$\Longrightarrow \boxed{x_i = \frac{\text{det.}A \text{ (with replacing the column } i \text{ with } b)}{\text{det.}A}}$$

Cramer's Rule.

Ex.1 : Solve $3x - y = 9$

$x + 2y = -4$

sol. $\det.A = \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} = 6 + 1 = 7$

$$x = \frac{\begin{vmatrix} 9 & -1 \\ -4 & 2 \end{vmatrix}}{7} = \frac{18-4}{7} = \frac{14}{7} = 2$$

$$y = \frac{\begin{vmatrix} 3 & 9 \\ 1 & -4 \end{vmatrix}}{7} = \frac{-12-9}{7} = \frac{-21}{7} = -3$$

Values of x & y must satisfy the two equations.

Ex.2 Solve $2x + 3y + z = 2$

$-x + 2y + 3z = -1$

$-3x - 3y + z = 0$

sol.

$$\det.A = \begin{vmatrix} 2 & 3 & 1 \\ -1 & 2 & 3 \\ -3 & -3 & 1 \end{vmatrix} = 2(11) + 1(6) - 3(7) = 7$$

$$x = \frac{\begin{vmatrix} 2 & 3 & 1 \\ -1 & 2 & 3 \\ 0 & -3 & 1 \end{vmatrix}}{7} = \frac{2(11) + 1(6)}{7} = \frac{28}{7} = 4$$

$$y = \frac{\begin{vmatrix} 2 & 2 & 1 \\ -1 & -1 & 3 \\ -3 & 0 & 1 \end{vmatrix}}{7} = \frac{2(-1) + 1(2) - 3(7)}{7} = \frac{-21}{7} = -3$$

$$z = \frac{\begin{vmatrix} 2 & 3 & 2 \\ -1 & 2 & -1 \\ -3 & -3 & 0 \end{vmatrix}}{7} = \frac{2(9) + 1(3)}{7} = \frac{21}{7} = 3$$

Values of x, y & z must satisfy the three equations.

Useful facts about determinants:

① If two rows (or columns) of a matrix are identical the determinant is zero.

Ex. $A = \begin{bmatrix} 2 & 1 & 4 \\ 3 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix} \Rightarrow \det. A = 0$

because 1st & 3rd row are identical

② Interchanging two rows (or columns) of a matrix changes the sign of the determinant.

Ex. $A = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} \Rightarrow \det. A = (2 * -1) - (3 * 1) = -5$

$B = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \Rightarrow \det. B = 5$

[1st column with 2nd column]

③ $\det. A^T = \det. A$

④ Multiplying each element of a row (or column) by constant C multiplies the determinant by C.

Ex. $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \xrightarrow{1^{st} \text{ row} * 3} B = \begin{bmatrix} 3(1) & 3(2) \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 3 & -1 \end{bmatrix}$

$\det. A = -7 \Longrightarrow \det. B = 3 * -7 = -21$

⑤ If all elements of a matrix above (or below) the main diagonal are zeroes, then the determinant of the matrix is the product of the elements on the main diagonal.

Ex. $A = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix} \Longrightarrow \det. A = 3 * 1 * 2 = 6$